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The maximal $D = 4$ supergravities

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Abstract: All maximal supergravities in four space-time dimensions are presented. The ungauged Lagrangians can be encoded in an $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$ matrix associated with the freedom of performing electric/magnetic duality transformations. The gauging is defined in terms of an embedding tensor Θ which encodes the subgroup of $\mathrm{E}_{7(7)}$ that is realized as a local invariance. This embedding tensor may imply the presence of magnetic charges which require corresponding dual gauge fields. The latter can be incorporated by using a recently proposed formulation that involves tensor gauge fields in the adjoint representation of $\mathrm{E}_{7(7)}$. In this formulation the results take a universal form irrespective of the electric/magnetic duality basis. We present the general class of supersymmetric and gauge invariant Lagrangians and discuss a number of applications.

Keywords: Supergravity Models, Extended Supersymmetry, Gauge Symmetry, Flux compactifications.

Contents

1. Introduction

Maximal supergravity in four space-time dimensions contains 28 vector gauge fields, which, in principle, can couple to charges assigned to the various fields. To preserve supersymmetry these gauge field interactions must be accompanied by masslike terms for the fermions and a scalar potential, as was first exhibited in the gauging of SO(8) [1]. In general it is far from obvious which gauge groups are admissible and will lead to a supersymmetric deformation of the ungauged Lagrangian. Initially non-compact versions and/or contractions of SO(8) were shown to also lead to viable gaugings [2], followed, much more recently, by the so-called 'flat' gauge groups [3] that one obtains upon Scherk-Schwarz reductions [4, 5] of higher-dimensional theories, as well as by several other non-semisimple groups [6].

In [7] we presented an *ab initio* analysis of all possible gaugings of four-dimensional maximal supergravity (this was reviewed in [8]). The gauge group, G_g, is a subgroup of the $E_{7(7)}$ duality group that leaves the combined field equations and Bianchi identities invariant. The decomposition of the gauge group generators in terms of the $E_{7(7)}$ generators is parametrized by the so-called embedding tensor Θ, which determines not only the gauge-covariant derivatives, but also the so-called T-tensors that define the masslike terms and the scalar potential. The admissible embedding tensors can be characterized group-theoretically and this enables a systematic discussion of all possible gaugings. In [7], several examples were presented which demonstrate how one can conveniently analyze various gaugings in this way. Another example, which is relevant for IIB flux compactifications, was worked out in [9]. The same strategy has been applied to maximal supergravity in various space-time dimensions [10 – 13], as well as to theories with a lower number of supercharges [14, 15].

In this paper we present a complete analysis of all gaugings of maximally supersymmetric four-dimensional supergravity. We establish that a gauging is in fact completely characterized by the embedding tensor, which is subject to two constraints. One constraint, which is linear, follows from supersymmetry and implies that the embedding tensor belongs to the **912** representation of $\mathrm{E}_{7(7)}$. A second constraint is quadratic and implies that the square of the embedding tensor does not contain the $\mathbf{133} + \mathbf{8645}$ representation. This constraint ensures the closure of the gauge group. Furthermore it implies that the embedding tensor is gauge invariant, and it ensures that the charges can always be chosen in the electric subsector upon a suitable electric/magnetic duality transformation. In this approach one can establish the consistency of the gauging prior to evaluating the explicit Lagrangian. Any given embedding tensor that satisfies these two constraints, defines a consistent supersymmetric and gauge invariant Lagrangian. In fact, we will present universal expressions for the Lagrangian and the supersymmetry transformations of gauged $N = 8$ supergravities, encoded in terms of the embedding tensor. The fermionic masslike terms and the scalar potential have a unique structure in terms of the so-called T-tensor, which is linearly proportional to the embedding tensor. Here we should perhaps emphasize that we our results are obtained entirely in a four-dimensional setting. As is well known, gaugings can originate from the compactification of a higher-dimensional theory with or without fluxes, or from Scherk-Schwarz reductions. But whatever their origin, the four-dimensional truncations belong to the class of theories discussed in this paper, provided they are maximally supersymmetric (irrespective of whether the theory will have maximally supersymmetric groundstates).

A gauging can involve both magnetic and electric charges, each of which will require corresponding gauge fields. These can be accommodated by making use of a new formalism [16], which, in the case at hand, requires the presence of tensor gauge fields transforming in the (adjoint) **133** representation of $\mathrm{E}_{7(7)}$. Neither the magnetic vector fields nor the tensor fields lead to additional degrees of freedom owing to the presence of extra gauge invariances associated with these fields. Because of the extra fields, any embedding tensor that satisfies the above constraints will lead to a consistent gauge invariant and supersymmetric theory, irrespective of whether the charges are electric or magnetic.

There are two characteristic features that play an important role in this paper. One that is typical of four-dimensional theories with vector gauge fields, concerns electric/magnetic duality [17]. For zero gauge-coupling constant, the gauge fields transform in the **56** representation of $\mathrm{E}_{7(7)}$, and decompose into 28 electric gauge fields and their 28 magnetic duals. The magnetic duals do not appear in the Lagrangian, so that the

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Lagrangian cannot be invariant under $E_{7(7)}$, but the combined equations of motion and Bianchi identities of the vector fields do transform covariantly in the **56** representation [18]. In fact the rigid symmetry group of the Lagrangian is a subgroup of $E_{7(7)}$ under whose action electric gauge fields are transformed into electric gauge fields. This group is not unique. It depends on the embedding of $E_{7(7)}$ inside the larger duality group $\mathrm{Sp}(56;\mathbb{R})$, which determines which gauge fields belonging to the **56** representation play the role of electric and which ones the role of magnetic gauge fields. The choice of the electric/magnetic frame fixes the rigid symmetry group of the ungauged Lagrangian, and different choices yield in general different Lagrangians which are not related to each other by local field redefinitions.

The conventional approach for introducing local gauge invariance relies on the rigid symmetry group of the ungauged Lagrangian as the gauge group has to be a subgroup thereof. The procedure requires the introduction of minimal couplings involving only the electric vector fields, and therefore it explicitly breaks the original $E_{7(7)}$ duality covariance of the field equations and Bianchi identities. The advantage of the formulation proposed in [16] is two-fold. On the one hand, minimal couplings involve both electric and magnetic vector fields in symplectically invariant combinations with the corresponding components of the embedding tensor. This ensures that, irrespective of the gauge group, the $E_{7(7)}$ invariance can formally be restored at the level of the field equations and Bianchi identities, provided the embedding tensor is treated as a "spurionic" object transforming under $E_{7(7)}$ and subject to the two aforementioned group-theoretical constraints. On the other hand, we are no longer restricted in the choice of the gauge group by the rigid symmetries of the original ungauged Lagrangian. Regardless of the electric/magnetic frame, we may introduce any gauge group contained in $E_{7(7)}$ corresponding to an embedding tensor that satisfies the two $E_{7(7)}$-covariant constraints. If this gauge group is not a subgroup of the rigid symmetry group of the ungauged Lagrangian, the embedding tensor will typically lead to magnetic charges and magnetic gauge fields together with the tensor fields. The latter will play a crucial role in realizing the gauge invariance of the final Lagrangian. An interesting feature of the resulting theory is that the scalar potential is described by means of a universal formula which is independent of the electric/magnetic duality frame.

A second, more general, feature of maximal supergravity is that the scalar fields parametrize a symmetric space, in this case the coset space $E_{7(7)}/\mathrm{SU}(8)$. The standard treatment of the corresponding gauged nonlinear sigma models is based on a formulation in which the group SU(8) is realized as a local invariance which acts on the spinor fields and the scalars; the corresponding connections are composite fields. A gauging is based on a group $G_g \subset E_{7(7)}$ whose connections are provided by (some of the) elementary vector gauge fields of the supergravity theory. The coupling constant associated with the gauge group will be denoted by g. One can impose a gauge condition with respect to the local SU(8) invariance which amounts to fixing a coset representative for the coset space. In that case the $E_{7(7)}$-symmetries will act nonlinearly on the fields and these nonlinearities make many calculations intractable. Because it is much more convenient to work with symmetries that are realized linearly, the best strategy is therefore to postpone this gauge fixing until the end. This strategy was already adopted in [1], but in this paper we find it convenient to introduce a slightly different definition of the coset representative.

Let us end this introduction by making some remarks on the physical significance of the embedding tensor. As previously anticipated, the low-energy dynamics of any superstring/M-theory compactification that admits a four dimensional $N = 8$ effective supergravity description, has to be contained within the class of theories discussed in the present paper. From the higher-dimensional perspective a gauging is in general characterized by constant background quantities which may be related to fluxes of higher-dimensional field strengths across cycles of the compactification manifold (form-fluxes), or just associated with the geometry of the internal manifold (geometric-fluxes), such as the tensor defining a twist in the topology in an internal torus [4]. In all known instances of gauged extended supergravities arising from superstring/M-theory compactifications, these background quantities enter the four-dimensional theory as components of the embedding tensor. Interestingly enough, in these cases the quadratic constraint on the embedding tensor follows from consistency of the higher-dimensional field equations and Bianchi identities. For instance, in type-II compactifications in the presence of form-fluxes the quadratic constraint expresses the tadpole cancellation condition. This condition, in the context of compactifications which are effectively described by $N = 8$ four-dimensional supergravity, poses severe restrictions on the fluxes since there is no room in this framework for localized sources such as orientifold planes. This is the case, for example, for the type-IIB theory compactified on a six-torus in the presence of NS-NS and R-R form-fluxes. The situation is clearly different for compactifications yielding $N \leq 4$ theories in four dimensions.

Having identified the background quantities in a generic flux compactification with components of the embedding tensor, our formulation of gauged maximal supergravity may provide a useful setting for studying the duality relations between more general superstring/M-theory vacua. Indeed the embedding tensor transforms covariantly with respect to the full rigid symmetry group $\mathrm{E}_{7(7)}$ of the four-dimensional theory, which is expected to encode the various string dualities. For instance, the generic T-duality transformations on the string moduli of the six-torus, within the same type-II theory, are implemented by the $\mathrm{SO}(6,6;\mathbb{Z})$ subgroup of $\mathrm{E}_{7(7)}$.

This paper is organized as follows. In section 2 the embedding tensor is introduced together with an extensive discussion of the constraints it should satisfy. It is demonstrated in a special electric/magnetic frame how these constraints ensure the existence of a Lagrangian that is invariant under the gauge group specified by the embedding tensor. Furthermore it is explained how to incorporate both electric and magnetic charges and corresponding gauge fields. In section 3 the corresponding T-tensor is introduced. As a result of the constraints on the embedding tensor the T-tensor satisfies a number of identities which are important for the supersymmetry of the Lagrangian. In section 4 the Lagrangian and the supersymmetry transformations are derived. Salient features are the universal expressions for the fermionic masslike terms and the scalar potential, which are induced by the gauging, as well as the role played by the magnetic gauge fields. Some applications, including explicit examples of new gaugings, are reviewed in section 5.

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2. The embedding tensor

We start by considering (abelian) vector fields ${A_\mu}^M$ transforming in the **56** representation of the $\mathrm{E}_{7(7)}$ duality group with generators denoted by $(t_\alpha)_M{}^N$, so that $\delta {A_\mu}^M = -\Lambda^\alpha (t_\alpha)_N{}^M \, {A_\mu}^N$. These vector potentials can be decomposed into 28 electric potentials ${A_\mu}^\Lambda$ and 28 magnetic potentials $A_{\mu\Lambda}$. In the conventional supergravity Lagrangians only 28 electric vectors appear, but at this stage we base ourselves on 56 gauge fields. In due course we will see how the correct balance of physical degrees of freedom is nevertheless realized. The gauge group must be a subgroup of $\mathrm{E}_{7(7)}$, so that its generators X_M, which couple to the gauge fields ${A_\mu}^M$, are decomposed in terms of the 133 independent $\mathrm{E}_{7(7)}$ generators t_α, i.e.,

$$X_M = \Theta_M{}^\alpha \, t_\alpha \,, \tag{2.1}$$

where $\alpha = 1, 2, \ldots, 133$ and $M = 1, 2, \ldots, 56$. The gauging is thus encoded in a real *embedding tensor* $\Theta_M{}^\alpha$ belonging to the $\mathbf{56} \times \mathbf{133}$ representation of $\mathrm{E}_{7(7)}$. The embedding tensor acts as a projector whose rank r equals the dimension of the gauge group. One expects that $r \leq 28$, because the ungauged Lagrangian should be based on 28 vector fields to describe the physical degrees of freedom. As we shall see shortly, this bound is indeed satisfied. The strategy of this paper is to treat the embedding tensor as a spurionic object that transforms under the duality group, so that the Lagrangian and transformation rules remain formally invariant under $\mathrm{E}_{7(7)}$. The embedding tensor can then be characterized group-theoretically. When freezing $\Theta_M{}^\alpha$ to a constant, the $\mathrm{E}_{7(7)}$-invariance is broken. An admissible embedding tensor is subject to a linear and a quadratic constraint, which ensure that one is dealing with a proper subgroup of $\mathrm{E}_{7(7)}$ and that the corresponding supergravity action remains supersymmetric. These constraints are derived in the first subsection. A second subsection elucidates some of the results in a convenient $\mathrm{E}_{7(7)}$ basis. A third subsection deals with the introduction of tensor gauge fields and their relevance for magnetic charges.

2.1 The constraints on the embedding tensor

The fact that the X_M generate a group and thus define a Lie algebra,

$$[X_M, X_N] = f_{MN}{}^P \, X_P, \tag{2.2}$$

with $f_{MN}{}^P$ the as yet unknown structure constants of the gauge group, implies that the embedding tensor must satisfy the closure condition,

$$\Theta_M{}^\alpha \, \Theta_N{}^\beta \, f_{\alpha\beta}{}^\gamma = f_{MN}{}^P \, \Theta_P{}^\gamma \,. \tag{2.3}$$

Here the $f_{\alpha\beta}{}^\gamma$ denote the structure constants of $\mathrm{E}_{7(7)}$, according to $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma \, t_\gamma$. The closure condition implies that the structure constants $f_{MN}{}^P$ satisfy the Jacobi identities in the subspace projected by the embedding tensor,

$$f_{[MN}{}^Q \, f_{P]Q}{}^R \, \Theta_R{}^\alpha = 0 \,. \tag{2.4}$$

Using the gauge group generators X_M one introduces gauge covariant derivatives,

$$D_\mu = \partial_\mu - g\, A_\mu{}^M\, X_M\,, \tag{2.5}$$

where g denotes an uniform gauge coupling constant. These derivatives lead to the covariant field strengths,

$$\Theta_M{}^\alpha\, \mathcal{F}_{\mu\nu}{}^M = \Theta_M{}^\alpha\, (\partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M - g\, f_{NP}{}^M\, A_\mu{}^N A_\nu{}^P)\,. \tag{2.6}$$

The gauge field transformations are given by

$$\Theta_M{}^\alpha\, \delta A_\mu{}^M = \Theta_M{}^\alpha\, (\partial_\mu \Lambda^M - g\, f_{NP}{}^M\, A_\mu{}^N\, \Lambda^P)\,. \tag{2.7}$$

Because of the contraction with the embedding tensor, the above results apply only to an r-dimensional subset of the gauge fields; the remaining ones do not appear in the covariant derivatives and are not directly involved in the gauging. However, the r gauge fields that do appear in the covariant derivatives, are only determined up to additive terms linear in the $56 - r$ gauge fields that vanish upon contraction with $\Theta_M{}^\alpha$.

While the gauge generators (2.1) act in principle uniformly on all fields that transform under $\mathrm{E}_{7(7)}$, the gauge field transformations are a bit more subtle to determine. This is so because the gauge fields involved in the gauging should transform in the adjoint representation of the gauge group. At the same time their charges should coincide with X_M in the **56** representation, so that $(X_M)_N{}^P$ must decompose into the adjoint representation of the gauge group plus possible extra terms which vanish upon contraction with the embedding tensor,

$$(X_M)_N{}^P\, \Theta_P{}^\alpha \equiv \Theta_M{}^\beta\, t_{\beta N}{}^P\, \Theta_P{}^\alpha = -f_{MN}{}^P\, \Theta_P{}^\alpha\,. \tag{2.8}$$

These extra terms, pertaining to the gauge fields that do not appear in the covariant derivatives, will be considered in due course. Note that (2.8) is the analogue of (2.3) in the **56** representation. The combined conditions (2.3) and (2.8) imply that Θ is invariant under the gauge group and yield the $\mathrm{E}_{7(7)}$-covariant condition

$$C_{MN}{}^\alpha \equiv f_{\beta\gamma}{}^\alpha\, \Theta_M{}^\beta\, \Theta_N{}^\gamma + t_{\beta N}{}^P\, \Theta_M{}^\beta\, \Theta_P{}^\alpha = 0\,. \tag{2.9}$$

Obviously $C_{MN}{}^\alpha$ can be assigned to irreducible $\mathrm{E}_{7(7)}$ representations contained in the $\mathbf{56} \times \mathbf{56} \times \mathbf{133}$ representation. The condition (2.9) encompasses all previous results: it implies that

$$[X_M, X_N] = -X_{MN}{}^P\, X_P, \tag{2.10}$$

so that (2.9) implies a closed gauge algebra, whose structure constants, related to $X_{MN}{}^P$ in accord with (2.8), have the required antisymmetry. Hence (2.9) is indeed sufficient for defining a proper subgroup embedding.[1]

[1]Note that for an abelian gauge group we have $X_{MN}{}^P\Theta_P{}^\alpha = 0$. Using (2.12) this leads to $\mathrm{tr}(X_M\, X_N) = 0$.

The embedding tensor satisfies a second constraint, which is required by supersymmetry. This constraint is linear and amounts to restricting $\Theta_M{}^\alpha$ to the **912** representation [7]. From

$$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480}\,, \tag{2.11}$$

one shows that this condition on the representation implies the equations,

$$t_{\alpha M}{}^N\,\Theta_N{}^\alpha = 0\,, \qquad (t_\beta t^\alpha)_M{}^N\,\Theta_N{}^\beta = -\frac{1}{2}\,\Theta_M{}^\alpha\,, \tag{2.12}$$

where the index α is raised by the inverse of the $\mathrm{E}_{7(7)}$-invariant metric $\eta_{\alpha\beta} = \mathrm{tr}(t_\alpha t_\beta)$.

As a result of the representation constraint, the representation content of $C_{MN}{}^\alpha$ can be further restricted as from (2.12) one can derive the following equations,

$$t_{\alpha N}{}^P\,C_{MP}{}^\alpha = 0\,, \quad (t_\beta\,t^\alpha)_N{}^P\,C_{MP}{}^\beta = -\frac{1}{2}\,C_{MN}{}^\alpha\,, \quad t_{\alpha M}{}^P\,C_{PN}{}^\alpha = t_{\alpha N}{}^P\,C_{PM}{}^\alpha\,. \tag{2.13}$$

They imply that $C_{MN}{}^\alpha$ should belong to representations contained in $\mathbf{56}\times\mathbf{912}$. On the other hand, the product of two Θ-tensors belongs to the symmetric product of two **912** representations. Comparing the decomposition of these two products,[2]

$$\begin{aligned}(\mathbf{912}\times\mathbf{912})_{\mathrm{s}} &= \mathbf{133} + \mathbf{8645} + \mathbf{1463} + \mathbf{152152} + \mathbf{253935}\,,\\ \mathbf{56}\times\mathbf{912} &= \mathbf{133} + \mathbf{8645} + \mathbf{1539} + \mathbf{40755}\,,\end{aligned} \tag{2.14}$$

one deduces that $C_{MN}{}^\alpha$ belongs to the $\mathbf{133}+\mathbf{8645}$ representation. Noting the decomposition $(\mathbf{133}\times\mathbf{133})_{\mathrm{a}} = \mathbf{133}+\mathbf{8645}$, we observe that there is an alternative way to construct these two representations which makes use of the fact that $\mathrm{Sp}(56;\mathbb{R})$, and thus its $\mathrm{E}_{7(7)}$ subgroup, has an invariant skew-symmetric matrix Ω^{MN}, which we write as,

$$\Omega^{MN} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\,. \tag{2.15}$$

The conjugate matrix Ω_{MN} takes the same form, so that $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. In this way one derives an equivalent version of the constraint (2.9),

$$\Theta_M{}^\alpha\Theta_N{}^\beta\,\Omega^{MN} = 0 \iff \Theta^{\Lambda\,[\alpha}\Theta_\Lambda{}^{\beta]} = 0\,, \tag{2.16}$$

which is only equivalent provided the representation constraint (2.12) is imposed. The constraint (2.16) implies that the Θ^α can all be chosen as *electric* vectors upon a suitable $\mathrm{Sp}(56;\mathbb{R})$ transformation, implying that all the nonzero components of the 133 vectors Θ^α cover an r-dimensional subspace parametrized by the gauge fields $A_\mu{}^M$ with $M = 1,\ldots,r$ and $r \leq 28$. In this basis $X_{MN}{}^P$ can be written in triangular form,

$$X_M = \begin{pmatrix} -f_M & a_M \\ 0 & b_M \end{pmatrix}\,, \tag{2.17}$$

[2]We used the LiE package [19] for computing the decompositions of tensor products and the branching of representations.

where the $r\times r$ upper-left diagonal block coincides with the gauge group structure constants and the submatrices a_M and b_M do not contribute to the product $(X_M)_N{}^P\Theta_P{}^\alpha$. The lower-left $(56-r)\times r$ block vanishes as a result of (2.8). It is easy to see that a_M and b_M cannot both be zero. If that were the case, we would have $f_{MN}{}^P = -X_{MN}{}^P$, which is antisymmetric in M and N. Hence,

$$\Theta_N{}^\alpha\, t_{\alpha M}{}^P = -\Theta_M{}^\alpha\, t_{\alpha N}{}^P\,. \tag{2.18}$$

Contracting this result by $(t^\beta)_P{}^M$ leads to $t_\alpha\, t^\beta\, \Theta^\alpha = -\Theta^\beta$ which is in contradiction with the representation constraint (2.12). In the next subsection we give a more detailed analysis of the submatrices a_M and b_M, which shows that b_M never vanishes.

Let us now proceed and find the restrictions on $X_{MN}{}^P$. First of all, $\mathrm{E}_{7(7)}$ invariance of Ω^{MN} implies that $X_{MNP} = X_{MN}{}^Q\Omega_{PQ}$ is symmetric in N and P. Furthermore, X belongs to the **912** representation (remember that $(t_\alpha)_M{}^N$ transforms as an $\mathrm{E}_{7(7)}$ invariant tensor, so that X_M transforms in the same representation as the embedding tensor), which is, however, *not* contained in the symmetric product $(\mathbf{56}\times\mathbf{56}\times\mathbf{56})_\mathrm{s}$. Consequently it follows that the fully symmetric part of X_{MNP} must vanish. Likewise, contractions of $X_{MN}{}^P$ will also vanish, as they do not correspond to the **912** representation. Hence $X_{MN}{}^P$ has the following properties,

$$X_{M[NP]} = 0\,, \qquad X_{(MNP)} = 0\,, \qquad X_{MN}{}^N = X_{MN}{}^M = 0\,. \tag{2.19}$$

The first condition implies that

$$X_{M\Lambda}{}^\Sigma = -X_M{}^\Lambda{}_\Sigma\,, \quad X_{M\Lambda\Sigma} = X_{M\Sigma\Lambda}\,, \quad X_M{}^{\Lambda\Sigma} = X_M{}^{\Sigma\Lambda}\,, \tag{2.20}$$

whereas the second one implies

$$\begin{aligned} X^{(\Lambda\Sigma\Gamma)} &= 0\,, & 2X^{(\Gamma\Lambda)}{}_\Sigma &= X_\Sigma{}^{\Lambda\Gamma}\,,\\ X_{(\Lambda\Sigma\Gamma)} &= 0\,, & 2X_{(\Gamma\Lambda)}{}^\Sigma &= X^\Sigma{}_{\Lambda\Gamma}\,. \end{aligned} \tag{2.21}$$

The constraints (2.20) and (2.21) coincide with the constraints that we have adopted in a more general four-dimensional context in [16].

The constraint (2.16) motivates the definition of another tensor $Z^{M,\alpha}$, which is orthogonal to the embedding tensor, i.e. $Z^{M,\alpha}\Theta_M{}^\beta = 0$,

$$Z^{M,\alpha} \equiv \frac{1}{2}\Omega^{MN}\,\Theta_N{}^\alpha \quad\Longrightarrow\quad \begin{cases} Z^{\Lambda\alpha} = \ \ \tfrac{1}{2}\Theta^{\Lambda\alpha}\,,\\ Z_\Lambda{}^\alpha = -\tfrac{1}{2}\Theta_\Lambda{}^\alpha\,. \end{cases} \tag{2.22}$$

As a consequence of the second equation of (2.19), one may derive,

$$X_{(MN)}{}^P = Z^{P,\alpha}\, d_{\alpha\, MN}\,, \tag{2.23}$$

where $d_{\alpha\, MN}$ is an $\mathrm{E}_{7(7)}$-invariant tensor symmetric in (MN),

$$d_{\alpha\, MN} \equiv (t_\alpha)_M{}^P\,\Omega_{NP}\,. \tag{2.24}$$

The more general significance of (2.23) was discussed in [20].

2.2 A special $\mathrm{E}_{7(7)}$ basis

To appreciate the various implications of the constraints on $X_{MN}{}^{P}$, we consider a special basis in which all the charges are electric. Hence magnetic charges vanish by virtue of $\Theta^{\Lambda\alpha} = 0$. A vector V^M in the **56** representation can then be decomposed according to

$$V^M \longrightarrow (V^\Lambda, V_\Lambda) \longrightarrow (V^A, V^a, V_A, V_a) \,, \tag{2.25}$$

with $A = 1, \ldots, r$ and $a = r+1, \ldots, 28$; i.e., electric (gauge field) components are written with upper indices A, a and their magnetic duals with corresponding lower indices A, a. The components V^a then span the subspace defined by the condition $\Theta_\Lambda{}^\alpha\, V^\Lambda = 0$. Consequently, V^A and V_A are defined up to terms proportional to the V^a and V_a, respectively. Obviously only the $\Theta_A{}^\alpha$ are nonvanishing and the $X_{MN}{}^P$ are only nonzero when $M = A$. Imposing (2.23) and (2.24), it follows that a block decomposition of $X_{AN}{}^P$ is then as follows (row and column indices are denoted by B, b and C, c, respectively),

$$X_{AN}{}^P = \begin{pmatrix} -f_{AB}{}^C & h_{AB}{}^c & C_{ABC} & C_{ABc} \\ 0 & 0 & C_{ACb} & 0 \\ 0 & 0 & f_{AC}{}^B & 0 \\ 0 & 0 & -h_{AC}{}^b & 0 \end{pmatrix} , \tag{2.26}$$

where

$$h_{(AB)}{}^c = C_{(AB)c} = C_{A[BC]} = C_{(ABC)} = f_{(AB)}{}^C = f_{AB}{}^B = 0 \,. \tag{2.27}$$

The last equation implies that the gauge group is unimodular. The closure relations (2.2) imply a number of nontrivial identities,

$$\begin{aligned} f_{[AB}{}^D\, f_{C]D}{}^E &= 0 \,, \\ f_{[AB}{}^D\, h_{C]D}{}^a &= 0 \,, \\ f_{AB}{}^E\, C_{ECD} - 4\, f_{(C[A}{}^E\, C_{B]D)E} + 4\, h_{(C[A}{}^a\, C_{B]D)a} &= 0 \,, \\ f_{[AB}{}^D\, C_{C]Da} &= 0 \,. \end{aligned} \tag{2.28}$$

The transformations generated by (2.26) imply that electric gauge fields transform exclusively into electric gauge fields,

$$\begin{aligned} \delta A_\mu{}^A &= \Lambda^B\, f_{BC}{}^A\, A_\mu{}^C \,, \\ \delta A_\mu{}^a &= -\Lambda^B\, h_{BC}{}^a\, A_\mu{}^C \,, \end{aligned} \tag{2.29}$$

where, for the moment, we keep the transformation parameters Λ^A space-time independent. The magnetic gauge fields, on the other hand, transform into electric and magnetic gauge fields,

$$\begin{aligned} \delta A_{\mu A} &= -\Lambda^B (f_{BA}{}^C\, A_{\mu C} - h_{BA}{}^c\, A_{\mu c} + C_{BAC}\, A_\mu{}^C + C_{BAc}\, A_\mu{}^c) \,, \\ \delta A_{\mu a} &= -\, \Lambda^B\, C_{BAa}\, A_\mu{}^A \,. \end{aligned} \tag{2.30}$$

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Because the Lagrangian does not contain the magnetic gauge fields in this case, the question arises how the gauge transformations are realized. The answer is provided by electric/magnetic duality. The above variations (2.29) and (2.30) generate a subgroup of these duality transformations that must be contained in $\mathrm{E}_{7(7)}$. General electric/magnetic transformations constitute an even bigger group $\mathrm{Sp}(56;\mathbb{R})$. In the abelian case they are defined by rotations of the 28 field strengths $F_{\mu\nu}{}^{\Lambda}$ and the 28 conjugate tensors $G_{\mu\nu\,\Lambda}$ defined by

$$G_{\mu\nu\,\Lambda} = \mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}{}^{\Lambda}}\,. \tag{2.31}$$

The corresponding field equations and Bianchi identities constitute 56 equations,

$$\partial_{[\mu}F_{\nu\rho]}{}^{\Lambda} = 0 = \partial_{[\mu}G_{\nu\rho]\,\Lambda}\,, \tag{2.32}$$

which are clearly invariant under rotations of the 56 field strengths $G_{\mu\nu}{}^{M}$, defined by

$$G_{\mu\nu}{}^{M} \equiv \begin{pmatrix} F_{\mu\nu}{}^{\Lambda} \\ G_{\mu\nu\Sigma} \end{pmatrix}\,. \tag{2.33}$$

The equations (2.32) show that the $G_{\mu\nu}{}^{M}$ can be expressed in terms of 56 vector potentials, and this is how the electric and magnetic gauge fields appear in the abelian case. Hence we may write,

$$G_{\mu\nu}{}^{M} = 2\,\partial_{[\mu}A_{\nu]}{}^{M}\,. \tag{2.34}$$

Electric/magnetic duality acts in principle on (abelian) field strengths rather than on corresponding gauge fields, because the field strengths $G_{\mu\nu\Lambda}$ are not independent according to (2.31).

Let us briefly return to the general $\mathrm{Sp}(56;\mathbb{R})$ dualities, which can be decomposed as follows,

$$\begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} U^{\Lambda}{}_{\Sigma} & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_{\Lambda}{}^{\Sigma} \end{pmatrix} \begin{pmatrix} F^{\Sigma} \\ G_{\Sigma} \end{pmatrix}\,, \tag{2.35}$$

where the (real) constant matrix leaves the skew-symmetric matrix Ω_{MN} invariant. This ensures that the new dual field strengths $G_{\mu\nu\Lambda}$ can again be written in the form (2.31) but with a different Lagrangian. These duality transformations thus define equivalence classes of Lagrangians that lead to the same field equations and Bianchi identities. They are generalizations of the duality transformations known from Maxwell theory, which rotate the electric and magnetic fields and inductions (for a review of electric/magnetic duality, see [17]). An $\mathrm{E}_{7(7)}$ subgroup of these transformations, combined with transformations on the scalar fields, constitutes an *invariance* group, meaning that the combined field equations and Bianchi identities (including the field equations for the other fields) before and after the $\mathrm{E}_{7(7)}$ transformation follow from an identical Lagrangian. Only the vector field strengths (2.33) and the scalar fields (to be introduced in section 3) are subject to these $\mathrm{E}_{7(7)}$ transformations. The other fields, such as the vierbein field and the spinor fields, are inert under $\mathrm{E}_{7(7)}$.

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To be more specific let us introduce the generic gauge field Lagrangian that is at most quadratic in the field strengths, parametrized as in [16],

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{vector}} = & -\frac{1}{4}\mathrm{i}\Big\{\mathcal{N}_{\Lambda\Sigma}\, F_{\mu\nu}^{+\,\Lambda}\, F^{+\mu\nu\Sigma} - \bar{\mathcal{N}}_{\Lambda\Sigma}\, F_{\mu\nu}^{-\,\Lambda}\, F^{-\mu\nu\Sigma}\Big\} \\ & + F_{\mu\nu}^{+\Lambda}\, \mathcal{O}_{\Lambda}^{+\mu\nu} + F_{\mu\nu}^{-\Lambda}\, \mathcal{O}_{\Lambda}^{-\mu\nu} \\ & + \mathrm{i}[(\mathcal{N}-\bar{\mathcal{N}})^{-1}]^{\Lambda\Sigma}\left[\mathcal{O}^{+}_{\mu\nu\Lambda}\mathcal{O}^{+\mu\nu}_{\Sigma} + \mathcal{O}^{-}_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}_{\Sigma}\right]. \end{aligned} \tag{2.36}$$

Here the $F_{\mu\nu}^{\pm}$ are complex (anti-)selfdual combinations normalized such that $F_{\mu\nu} = F_{\mu\nu}^{+} + F_{\mu\nu}^{-}$. The field-dependent symmetric tensor $\mathcal{N}_{\Lambda\Sigma}$ comprises the generalized theta angles and coupling constants and $\mathcal{O}^{\pm}_{\mu\nu\,\Lambda}$ represents bilinears in the fermion fields. The terms quadratic in ${\mathcal{O}_{\mu\nu}}^{\Lambda}$ are such that any additional terms in the Lagrangian (which no longer depends on the field strengths) will transform covariantly under electric/magnetic duality. From the above Lagrangian we derive

$$G^{+}_{\mu\nu\Lambda} = \mathcal{N}_{\Lambda\Sigma}\, F_{\mu\nu}^{+\,\Sigma} + 2\mathrm{i}\,\mathcal{O}^{+}_{\mu\nu\Lambda}\,. \tag{2.37}$$

Upon an electric/magnetic duality transformation (2.35) one finds an alternative Lagrangian of the same form but with a different expression for $\mathcal{N}_{\Lambda\Sigma}$ and $\mathcal{O}_{\Lambda}$,

$$\begin{aligned} \mathcal{N}_{\Lambda\Sigma} &\longrightarrow (V\mathcal{N}+W)_{\Lambda\Gamma}\,[(U+Z\mathcal{N})^{-1}]^{\Gamma}{}_{\Sigma}\,, \\ \mathcal{O}^{+}_{\mu\nu\,\Lambda} &\longrightarrow \mathcal{O}^{+}_{\mu\nu\,\Sigma}\,[(U+Z\mathcal{N})^{-1}]^{\Sigma}{}_{\Lambda}\,, \end{aligned} \tag{2.38}$$

This result follows from requiring consistency between (2.31) and (2.35). The restriction to $\mathrm{Sp}(56;\mathbb{R})$ ensures that the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ remains preserved. For the $\mathrm{E}_{7(7)}$ subgroup of invariances, the transformations (2.38) must be induced by corresponding $\mathrm{E}_{7(7)}$ transformations of the scalar fields.

Let us now return to the infinitesimal gauge transformations corresponding to the charges (2.26), which act on the field strengths according to $\delta {F_{\mu\nu}}^{M} = -\Lambda^{A}\, {X_{AN}}^{M}\, {F_{\mu\nu}}^{N}$. The abelian field strengths ${F_{\mu\nu}}^{\Lambda}$ and $G_{\mu\nu\Lambda}$ thus transform as

$$\begin{aligned} \delta {F_{\mu\nu}}^{A} &= \Lambda^{B}\, {f_{BC}}^{A}\, {F_{\mu\nu}}^{C}\,, \\ \delta {F_{\mu\nu}}^{a} &= -\Lambda^{B}\, {h_{BC}}^{a}\, {F_{\mu\nu}}^{C}\,, \\ \delta G_{\mu\nu\,A} &= -\Lambda^{B}({f_{BA}}^{C}\, G_{\mu\nu\,C} - {h_{BA}}^{c}\, G_{\mu\nu\,c} + C_{BAC}\, {F_{\mu\nu}}^{C} + C_{BAc}\, {F_{\mu\nu}}^{c})\,, \\ \delta G_{\mu\nu\,a} &= -\Lambda^{B}\, C_{BAa}\, {F_{\mu\nu}}^{A}\,. \end{aligned} \tag{2.39}$$

According to (2.37) the field strengths $G_{\mu\nu\Lambda}$ depend also on fields other than the vector fields, and in order to have an invariance, the transformations of the these fields should combine with the transformations of the vector fields to yield the above variations for the dual field strengths $G_{\mu\nu\Lambda}$. Therefore the gauge group must be a subgroup of $\mathrm{E}_{7(7)}$. In that case it follows that the transformations (2.39) for ${F_{\mu\nu}}^{\Lambda}$ and $G_{\mu\nu\Lambda}$ leave the Lagrangian (2.36) invariant, up to

$$\delta\mathcal{L} \propto \varepsilon^{\mu\nu\rho\sigma}\, \Lambda^{A}\left[C_{ABC}\, {F_{\mu\nu}}^{B}\, {F_{\rho\sigma}}^{C} + 2\, C_{ABa}\, {F_{\mu\nu}}^{B}\, {F_{\rho\sigma}}^{a}\right]. \tag{2.40}$$

This variation constitutes a total derivative when the Λ^A are constant. When the parameters Λ^A are space-time dependent, one needs to introduce extra terms into the Lagrangian. According to (2.26) the gauge fields transform as,

$$\begin{aligned} \delta A_\mu{}^A &= \partial_\mu \Lambda^A - g\, f_{BC}{}^A\, A_\mu{}^B\, \Lambda^C\,, \\ \delta A_\mu{}^a &= \partial_\mu \Lambda^a + g\, h_{BC}{}^a\, A_\mu{}^B\, \Lambda^C\,, \end{aligned} \tag{2.41}$$

and the covariant field strengths acquire the standard non-abelian modifications,

$$\begin{aligned} F_{\mu\nu}{}^A \to \mathcal{F}_{\mu\nu}{}^A &= \partial_\mu A_\nu{}^A - \partial_\nu A_\mu{}^A - g\, f_{BC}{}^A\, A_\mu{}^B A_\nu{}^C\,, \\ F_{\mu\nu}{}^a \to \mathcal{F}_{\mu\nu}{}^a &= \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\, h_{BC}{}^a\, A_\mu{}^B A_\nu{}^C\,. \end{aligned} \tag{2.42}$$

Likewise the derivatives on the scalar fields are extended to properly covariantized derivatives according to (2.5). The only gauge fields that appear in the covariant derivatives are the fields $A_\mu{}^A$, so that only these gauge fields couple to the matter fields. Note that, according to (2.41) and (2.42), the abelian gauge fields $A_\mu{}^a$ couple to charges that are central in the gauge algebra. Therefore the resulting gauge algebra is a central extension of (2.2). Introducing formal generators $\tilde{X}_A$ and $\tilde{X}_a$, it reads,

$$[\tilde{X}_A, \tilde{X}_B] = f_{AB}{}^C\, \tilde{X}_C - h_{AB}{}^a \tilde{X}_a\,. \tag{2.43}$$

On the matter fields the central charges $\tilde{X}_a$ vanish and $\tilde{X}_A = X_A$.

In (2.40) the abelian field strengths will be replaced by the covariant field strenths (2.42), so that (2.40) is no longer a total derivative. Therefore the invariance of the action requires the presence of extra Chern-Simons-like terms,

$$\begin{aligned} \mathcal{L}^{\rm CS} \propto g\, \varepsilon^{\mu\nu\rho\sigma} \Big[& C_{ABC}\, A_\mu{}^A A_\nu{}^B (\partial_\rho A_\sigma{}^C - \tfrac{3}{8} g\, f_{DE}{}^C A_\rho{}^D A_\sigma{}^E) \\ & + C_{ABa}\, A_\mu{}^A (A_\nu{}^B \partial_\rho A_\sigma{}^a + A_\nu{}^a \partial_\rho A_\sigma{}^B) \\ & + \tfrac{3}{8} g\, C_{ABa}\, A_\mu{}^A (h_{CD}{}^a A_\nu{}^B - f_{CD}{}^B\, A_\nu{}^a) A_\rho{}^C A_\sigma{}^D \Big]\,. \end{aligned} \tag{2.44}$$

The identities (2.28) ensure that these terms are indeed sufficient for restoring the gauge invariance of the Lagrangian [21, 22]. In this connection it is important that the definition of the dual field strengths remains as in (2.31), so that $\mathcal{G}_{\mu\nu\Lambda}$ will be defined by (2.37) with $F_{\mu\nu}{}^\Lambda$ replaced by the non-abelian field strengths $\mathcal{F}_{\mu\nu}{}^\Lambda$ defined in (2.42).

Hence we have shown that any embedding tensor that satisfies the two constraints (2.9) and (2.12), leads to a gauge invariant Lagrangian. We emphasize once more that this was done in the special basis (2.25), in which the charges are electric. The magnetic gauge fields do not play a role here and in the non-abelian case they can no longer be defined in terms of a solution of (2.34).

2.3 Magnetic potentials and antisymmetric tensor fields

In the more general setting with magnetic charges, the gauge algebra does not close, simply because the Jacobi identity is only valid on the subspace projected by the embedding tensor (c.f. (2.4)). As was generally proven in [16] for four-dimensional gauge theories, one can still obtain a consistent gauge algebra, provided one introduces magnetic gauge fields from the beginning, together with tensor gauge fields $B_{\mu\nu\alpha}$. In the case at hand these fields transform in the adjoint **133** representation of $\mathrm{E}_{7(7)}$. At the same time, to avoid unwanted degrees of freedom, the gauge transformations associated with the tensor fields should act on the (electric and magnetic) gauge fields by means of a transformation that also depends on the embedding tensor,

$$\delta A_\mu{}^M = D_\mu \Lambda^M - g\, Z^{M,\alpha}\, \Xi_{\mu\,\alpha}\,, \tag{2.45}$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $D_\mu\Lambda^M = \partial_\mu \Lambda^M + g\, X_{PQ}{}^M\, A_\mu{}^P \Lambda^Q$. The transformations proportional to $\Xi_{\mu\,\alpha}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $X_{MN}{}^P$ where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor). These gauge transformations form a group, as follows from the commutation relations,

$$\begin{aligned} [\delta(\Lambda_1), \delta(\Lambda_2)] &= \delta(\Lambda_3) + \delta(\Xi_3)\,, \\ [\delta(\Lambda), \delta(\Xi)] &= \delta(\tilde{\Xi})\,, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} \Lambda_3{}^M &= g\, X_{[NP]}{}^M \Lambda_2^N \Lambda_1^P\,, \\ \Xi_{3\mu\alpha} &= d_{\alpha NP} (\Lambda_1^N D_\mu \Lambda_2^P - \Lambda_2^N D_\mu \Lambda_1^P)\,, \\ \tilde{\Xi}_{\mu\alpha} &= g \Lambda^P (X_{P\alpha}{}^\beta + 2 d_{\alpha PN} Z^{N,\beta}) \Xi_{\mu\beta}\,. \end{aligned} \tag{2.47}$$

In order to write down invariant kinetic terms for the gauge fields we have to define a suitable covariant field strength tensor. This is an issue because the Jacobi identity is not satisfied and because we have to deal with the new gauge transformations parametrized by the parameters $\Xi_{\mu\alpha}$. Indeed, the usual field strength, which follows from the Ricci identity, $[D_\mu, D_\nu] = -g \mathcal{F}_{\mu\nu}{}^M\, X_M$,

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{[NP]}{}^M\, A_\mu{}^N A_\nu{}^P\,, \tag{2.48}$$

is not fully covariant.[3] The lack of covariance can be readily checked by observing that $\mathcal{F}_{\mu\nu}{}^M$ does not satisfy the Palatini identity,

$$\delta \mathcal{F}_{\mu\nu}{}^M = 2\, D_{[\mu} \delta A_{\nu]}{}^M - 2g\, X_{(PQ)}{}^M\, A_{[\mu}{}^P\, \delta A_{\nu]}{}^Q\,, \tag{2.49}$$

under arbitrary variations $\delta A_\mu{}^M$. This result shows that $\mathcal{F}_{\mu\nu}{}^M$ transforms under gauge transformations as

$$\delta \mathcal{F}_{\mu\nu}{}^M = g\, \Lambda^P X_{NP}{}^M\, \mathcal{F}_{\mu\nu}{}^N - 2g\, Z^{M,\alpha} (D_{[\mu} \Xi_{\nu]\alpha} + d_{\alpha PQ}\, A_{[\mu}{}^P\, \delta A_{\nu]}{}^Q)\,, \tag{2.50}$$

[3] Observe that the covariant derivative is invariant under the tensor gauge transformations, so that the field strengths contracted with X_M are in fact covariant.

which is not covariant. The standard strategy [11, 20, 16] is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}{}^{M} = \mathcal{F}_{\mu\nu}{}^{M} + g\, Z^{M,\alpha}\, B_{\mu\nu\,\alpha}\,, \tag{2.51}$$

where we introduce the tensor fields $B_{\mu\nu\,\alpha}$, which are subject to suitably chosen gauge transformation rules.

At this point we recall that the invariance transformations in the rigid case implied that the field strengths $G_{\mu\nu}{}^{M}$ transform under a subgroup of $\mathrm{Sp}(56,\mathbb{R})$ (c.f. (2.35)). Our aim is to find a similar symplectic array of field strengths so that these transformations are generated in the non-abelian case as well. This is not possible based on the variations of the vector fields $A_\mu{}^M$, which will never generate the type of fermionic terms contained in $G_{\mu\nu\Lambda}$. However, the presence of the tensor fields enables one to achieve this objective, at least to some extent. Just as in the abelian case, we define an $\mathrm{Sp}(56,\mathbb{R})$ array of field strengths $\mathcal{G}_{\mu\nu}{}^{M}$ by

$$\mathcal{G}_{\mu\nu}{}^{M} \equiv \begin{pmatrix} \mathcal{H}_{\mu\nu}{}^{\Lambda} \\ \mathcal{G}_{\mu\nu\Sigma} \end{pmatrix}\,, \tag{2.52}$$

so that

$$\begin{aligned} \mathcal{G}^{+}_{\mu\nu}{}^{\Lambda} &= \mathcal{H}^{+}_{\mu\nu}{}^{\Lambda}\,,\\ \mathcal{G}^{+}_{\mu\nu\Lambda} &= \mathcal{N}_{\Lambda\Sigma}\,\mathcal{H}^{+}_{\mu\nu}{}^{\Sigma} + 2\mathrm{i}\,\mathcal{O}^{+}_{\mu\nu\Lambda}\,. \end{aligned} \tag{2.53}$$

Note that the expression for $\mathcal{G}_{\mu\nu\Lambda}$ is the analogue of (2.37), with $F_{\mu\nu}{}^{\Lambda}$ replaced by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$.

Following [16] we introduce the following transformation rule for $B_{\mu\nu\alpha}$ (contracted with $Z^{M,\alpha}$, because only these combinations will appear in the Lagrangian),

$$Z^{M,\alpha}\delta B_{\mu\nu\,\alpha} = 2\, Z^{M,\alpha}(D_{[\mu}\Xi_{\nu]\alpha} + d_{\alpha\,NP}A_{[\mu}{}^{N}\delta A_{\nu]}{}^{P}) - 2\, X_{(NP)}{}^{M}\Lambda^{P}\mathcal{G}_{\mu\nu}{}^{N}\,, \tag{2.54}$$

where $D_\mu\Xi_{\nu\alpha} = \partial_\mu\Xi_{\nu\alpha} - gA_\mu{}^{M}X_{M\alpha}{}^{\beta}\Xi_{\nu\beta}$ with $X_{M\alpha}{}^{\beta} = -\Theta_M{}^{\gamma}f_{\gamma\alpha}{}^{\beta}$ the gauge group generators embedded in the adjoint representation of $\mathrm{E}_{7(7)}$. With this variation the modified field strengths (2.51) are invariant under tensor gauge transformations. Under the vector gauge transformations we derive the following result,

$$\begin{aligned} \delta\mathcal{G}^{+}_{\mu\nu}{}^{\Lambda} &= -g\,\Lambda^{P}X_{PN}{}^{\Lambda}\,\mathcal{G}^{+}_{\mu\nu}{}^{N} - g\,\Lambda^{P}X^{\Gamma}{}_{P}{}^{\Lambda}\,(\mathcal{G}^{+}_{\mu\nu} - \mathcal{H}^{+}_{\mu\nu})_{\Gamma}\,,\\ \delta\mathcal{G}^{+}_{\mu\nu\Lambda} &= -g\,\Lambda^{P}X_{PN\Lambda}\,\mathcal{G}^{+}_{\mu\nu}{}^{N} - g\,\mathcal{N}_{\Lambda\Sigma}\,\Lambda^{P}X^{\Gamma}{}_{P}{}^{\Sigma}\,(\mathcal{G}^{+}_{\mu\nu} - \mathcal{H}^{+}_{\mu\nu})_{\Gamma}\,,\\ \delta(\mathcal{G}^{+}_{\mu\nu} - \mathcal{H}^{+}_{\mu\nu})_{\Lambda} &= g\,\Lambda^{P}(X^{\Gamma}{}_{P\Lambda} - X^{\Gamma}{}_{P}{}^{\Sigma}\mathcal{N}_{\Sigma\Lambda})\,(\mathcal{G}^{+}_{\mu\nu} - \mathcal{H}^{+}_{\mu\nu})_{\Gamma}\,. \end{aligned} \tag{2.55}$$

Hence $\delta\mathcal{G}_{\mu\nu}{}^{M} = -g\,\Lambda^{P}X_{PN}{}^{M}\,\mathcal{G}_{\mu\nu}{}^{N}$, just as the variation of the abelian field strengths $G_{\mu\nu}{}^{M}$ in the absence of charges, up to terms proportional to $\Theta^{\Lambda\alpha}(\mathcal{G}_{\mu\nu} - \mathcal{H}_{\mu\nu})_{\Lambda}$. According to [16], the latter terms represent a set of field equations. The last equation of (2.55) then expresses the well-known fact that under a symmetry field equations transform into field equations. As a result the gauge algebra on these tensors closes according to (2.46), up to the the same field equation.

Having identified some of the field equations, it is easy to see how the Lagrangian should be modified. First of all, we replace the abelian field strengths $F_{\mu\nu}{}^{\Lambda}$ in the Lagrangian (2.36) by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, so that

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}_{\rm vector}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}}\,. \tag{2.56}$$

Under general variations of the vector and tensor fields we then obtain the result,

$$e^{-1}\delta\mathcal{L}_{\rm vector} = -\mathrm{i}\mathcal{G}^{+\mu\nu}{}_{\Lambda}\Big[D_\mu\delta A_\nu{}^{\Lambda} + \frac{1}{4}g\Theta^{\Lambda\alpha}(\delta B_{\mu\nu\alpha} - 2d_{\alpha PQ}A_\mu{}^{P}\delta A_\nu{}^{Q})\Big] + \text{h.c.}\,. \tag{2.57}$$

From this expression the reader can check that the Lagrangian (2.36) is indeed invariant under the tensor gauge transformations. Even when including the gauge transformations of the matter fields, the Lagrangian is, however, not invariant under the vector gauge transformations. For invariance it is necessary to introduce the following universal terms to the Lagrangian [16],

$$\begin{aligned}
\mathcal{L}_{\rm top} = {}& \frac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{\Lambda\alpha}\,B_{\mu\nu\,\alpha}\left(2\partial_\rho A_{\sigma\,\Lambda} + g\,X_{MN\,\Lambda}\,A_\rho{}^{M}A_\sigma{}^{N} + \frac{1}{4}g\,\Theta_{\Lambda}{}^{\beta}B_{\rho\sigma\,\beta}\right)\\
&+\frac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN\,\Lambda}\,A_\mu{}^{M}A_\nu{}^{N}\left(\partial_\rho A_\sigma{}^{\Lambda} + \frac{1}{4}g\,X_{PQ}{}^{\Lambda}A_\rho{}^{P}A_\sigma{}^{Q}\right)\\
&+\frac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN}{}^{\Lambda}\,A_\mu{}^{M}A_\nu{}^{N}\left(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4}g\,X_{PQ\Lambda}A_\rho{}^{P}A_\sigma{}^{Q}\right).
\end{aligned}\tag{2.58}$$

The first term represents a topological coupling of the antisymmetric tensor fields with the magnetic gauge fields, and the last two terms are a generalization of the Chern-Simons-like terms (2.44) that we encountered in the previous subsection. Under variations of the vector and tensor fields, this Lagrangian varies into (up to total derivative terms)

$$e^{-1}\delta\mathcal{L}_{\rm top} = \mathrm{i}\mathcal{H}^{+\mu\nu\Lambda}\,D_\mu\delta A_{\nu\Lambda} + \frac{1}{4}\mathrm{i}g\,\mathcal{H}^{+\mu\nu}{}_{\Lambda}\,\Theta^{\Lambda\alpha}(\delta B_{\mu\nu\alpha} - 2d_{\alpha PQ}A_\mu{}^{P}\delta A_\nu{}^{Q}) + \text{h.c.}\,. \tag{2.59}$$

Under the tensor gauge transformations this variation becomes equal to the real part of $2\mathrm{i}g\,\mathcal{H}^{+\mu\nu M}\,\Theta_M{}^{\alpha}\,D_\mu\Xi_{\nu\alpha}$. This expression equals a total derivative by virtue of the invariance of the embedding tensor, the constraint (2.16), and the Bianchi identity

$$D_{[\mu}\mathcal{H}_{\nu\rho]}{}^{M} = \frac{1}{3}g\,Z^{M,\alpha}\left[3\,D_{[\mu}B_{\nu\rho]\,\alpha} + 6\,d_{\alpha\,NP}\,A_{[\mu}{}^{N}(\partial_\nu A_{\rho]}{}^{P} + \frac{1}{3}gX_{[RS]}{}^{P}A_\nu{}^{R}A_{\rho]}{}^{S})\right]. \tag{2.60}$$

In this Bianchi identity, $D_\mu\mathcal{H}_{\nu\rho}{}^{M} = \partial_\mu\mathcal{H}_{\nu\rho}{}^{M} + gA_\mu{}^{P}X_{PN}{}^{M}\mathcal{H}_{\nu\rho}{}^{N}$ and $D_\rho B_{\mu\nu\alpha} = \partial_\rho B_{\mu\nu\alpha} - gA_\rho{}^{M}X_{M\alpha}{}^{\beta}B_{\mu\nu\beta}$. This expression for the Bianchi identity is suitable for our purpose here, but we note that it is not manifestly covariant in this form, in view of the fact that the fully covariant derivative of $\mathcal{H}_{\mu\nu}{}^{M}$ reads,

$$\mathcal{D}_\rho\mathcal{H}_{\mu\nu}{}^{M} = \partial_\rho\mathcal{H}_{\mu\nu}{}^{M} + gA_\rho{}^{P}X_{PN}{}^{M}\,\mathcal{G}_{\mu\nu}{}^{N} + gA_\rho{}^{P}X_{NP}{}^{M}\,(\mathcal{G}_{\mu\nu} - \mathcal{H}_{\mu\nu})^{N}\,, \tag{2.61}$$

and the covariant field strength of the tensor fields equals

$$\mathcal{H}_{\mu\nu\rho\,\alpha} \equiv 3\,D_{[\mu}B_{\nu\rho]\,\alpha} + 6\,d_{\alpha\,MN}\,A_{[\mu}{}^{M}\left(\partial_\nu A_{\rho]}{}^{N} + \frac{1}{3}gX_{[RS]}{}^{N}A_\nu{}^{R}A_{\rho]}{}^{S} + \mathcal{G}_{\nu\rho]}{}^{N} - \mathcal{H}_{\nu\rho]}{}^{N}\right). \tag{2.62}$$

The manifestly covariant form of the Bianchi identity (2.60) then reads,

$$\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho]}{}^{M} = \frac{1}{3}g\, Z^{M,\alpha}\, \mathcal{H}_{\mu\nu\rho\,\alpha}\, . \tag{2.63}$$

The various modifications described in this subsection ensure the gauge invariance of the Lagrangian $\mathcal{L}_{\rm vect} + \mathcal{L}_{\rm top}$, provided we include the gauge transformations of the scalar fields [16]. Furthermore, variation of the tensor fields yields the field equations identified above,

$$\delta\mathcal{L}_{\rm vector} + \delta\mathcal{L}_{\rm top} = -\frac{1}{4}{\rm i}g\;\delta B_{\mu\nu\alpha}\;\Theta^{\Lambda\alpha}\Big[(\mathcal{G}^{+\mu\nu} - \mathcal{H}^{+\mu\nu})_\Lambda - (\mathcal{G}^{-\mu\nu} - \mathcal{H}^{-\mu\nu})_\Lambda\Big]\, . \tag{2.64}$$

This result shows that the Lagrangian is invariant under variations of the tensor fields for those components that are projected to zero by the embedding tensor component $\Theta^{\Lambda\alpha}$. This implies that these components of the tensor field do not appear in the action, which plays a crucial role in ensuring that the number of degrees of freedom will remain unchanged.

A similar phenomenon takes place for the magnetic gauge fields $A_{\mu\Lambda}$. Evaluating the field equation for the gauge fields $A_\mu{}^M$ one finds that the equation for the magnetic gauge fields is only proportional to $\Theta^{\Lambda\alpha}\delta A_{\mu\Lambda}$. To see this, one evaluates

$$\delta\mathcal{L}_{\rm vector} + \delta\mathcal{L}_{\rm top} = \frac{1}{2}{\rm i}\,\varepsilon^{\mu\nu\rho\sigma}\; D_\nu\mathcal{G}_{\rho\sigma}{}^M\Omega_{MN}\delta A_\mu{}^N\, , \tag{2.65}$$

up to a total derivative and up to terms that vanish as a result of the field equation for $B_{\mu\nu\alpha}$. Here one makes use of (2.21). Note that $D_\nu\mathcal{G}_{\rho\sigma}{}^M = \mathcal{D}_\nu\mathcal{G}_{\rho\sigma}{}^M$, and furthermore that $\mathcal{D}_\nu\mathcal{G}_{\rho\sigma}{}^\Lambda = \mathcal{D}_\nu\mathcal{H}_{\rho\sigma}{}^\Lambda$, up to terms that vanish by virtue of the field equation for $B_{\mu\nu\alpha}$. Using the Bianchi identity (2.63) we can thus rewrite (2.65) as follows,

$$\delta\mathcal{L}_{\rm vector} + \delta\mathcal{L}_{\rm top} = \frac{1}{2}{\rm i}\,\varepsilon^{\mu\nu\rho\sigma}\left[-D_\nu\mathcal{G}_{\rho\sigma\Lambda}\,\delta A_\mu{}^\Lambda + \frac{1}{6}g\,\mathcal{H}_{\nu\rho\sigma\alpha}\,\Theta^{\Lambda\alpha}\delta A_{\mu\Lambda}\right]\, , \tag{2.66}$$

under the same conditions as stated above. Note that the minimal coupling of the gauge fields is always proportional to the embedding tensor. Therefore the full Lagrangian does not depend on those components of the magnetic gauge fields that are projected to zero by the embedding tensor component $\Theta^{\Lambda\alpha}$.

In the spririt of the analysis presented in [20], one may thus regard the absence of the components of $B_{\mu\nu\alpha}$ and $A_{\mu\Lambda}$ as resulting from an additional gauge invariance (which would then lead to rank-three tensors fields). However, since these fields will not appear in the Lagrangian, there is no need for doing so. Somewhat unexpectedly, and not in line with the general analysis of the vector-tensor hierarchies, there is an additional (local) invariance which involves only the tensor field [15],

$$\Theta^{\Lambda\alpha}\delta B_{\mu\nu\alpha} \propto \Delta^{\Lambda\Sigma\rho}{}_\rho\,(\mathcal{G}-\mathcal{H})_{\mu\nu\Sigma} - 6\,\Delta^{(\Lambda\Sigma)\rho}{}_{[\rho}\,(\mathcal{G}-\mathcal{H})_{\mu\nu]\Sigma}\, , \tag{2.67}$$

where $\Delta^{\Lambda\Sigma\mu}{}_\nu = \Theta^{\Lambda\alpha}\Delta_\alpha{}^{\Sigma\mu}{}_\nu$. This new invariance has, of course, a role to play in balancing the degrees of freedom, but in [16] this aspect was bypassed in the analysis. We note that not all of these gauge invariances have a bearing on the dynamic modes of the theory as they also act on fields that play an auxiliary role.

In spite of the modifications above, supersymmetry will be broken by the gauging. In section 4 we show how supersymmetry can be restored. But first we have to deal with the effect of the gauge transformations on the scalar fields.

3. The T-tensor

We already stressed in the introduction that the scalar fields parametrize the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset space.[4] These fields are described by a space-time dependent matrix $\mathcal{V}(x) \in \mathrm{E}_{7(7)}$ (taken in the fundamental **56** representation) which transforms from the right under local SU(8) and from the left under rigid $\mathrm{E}_{7(7)}$. The matrix $\mathcal{V}$ can be used to elevate the embedding tensor to the so-called T-tensor, which is the SU(8)-covariant, field-dependent, tensor that appears in the fermionic masslike terms and the scalar potential of the Lagrangian. The T-tensor is thus defined by,

$$T_{\underline{M}}{}^{\underline{\alpha}}[\Theta,\phi]\, t_{\underline{\alpha}} = \mathcal{V}^{-1}{}_{\underline{M}}{}^{N}\, \Theta_N{}^{\alpha}\, (\mathcal{V}^{-1} t_\alpha \mathcal{V}) \,, \tag{3.1}$$

where the underlined indices refer to local SU(8). The appropriate representation for (3.1) is the **56**, so that we may write,

$$T_{\underline{MN}}{}^{\underline{P}}[\Theta,\phi] = \mathcal{V}^{-1}{}_{\underline{M}}{}^{M}\, \mathcal{V}^{-1}{}_{\underline{N}}{}^{N}\, \mathcal{V}_P{}^{\underline{P}}\, X_{MN}{}^{P} \,. \tag{3.2}$$

Because the constraints on the embedding tensor are covariant under $\mathrm{E}_{7(7)}$, it is clear that they induce a corresponding set of SU(8) covariant constraints on the T-tensor.

However, we employ a somewhat unconventional definition of the coset representative $\mathcal{V}$. Note that the T-tensor is defined in an SU(8) covariant basis, where the maximal compact SU(8) subgroup of $\mathrm{E}_{7(7)}$ takes a block-diagonal form according to the branching under SU(8), $\mathbf{56} \to \mathbf{28} + \overline{\mathbf{28}}$. This implies the existence of a pseudo-real vector $U^{\underline{M}}$ decomposing according to $U^{\underline{M}} = (U^{ij}, U_{kl})$, where ij and kl denote antisymmetric index pairs with $i,j,k,l = 1,\ldots,8$. This basis facilitates the coupling to the fermions which transform under SU(8). On the other hand, just as in the preceding section, we decompose the gauge fields in a real basis according to $V^M = (V^\Lambda, V_\Sigma)$ which branches under the maximal real SL(8) subgroup of $\mathrm{E}_{7(7)}$ according to $\mathbf{56} \to \mathbf{28} + \mathbf{28}'$. Therefore we define 56-dimensional complex vectors $\mathcal{V}_M{}^{ij} = (\mathcal{V}_\Lambda{}^{ij}, \mathcal{V}^{\Sigma\, ij})$ and their complex conjugate $\mathcal{V}_{M\,ij} = (\mathcal{V}_{\Lambda\, ij}, \mathcal{V}^{\Sigma}{}_{ij})$, which together constitute a 56×56 matrix $\mathcal{V}$,

$$\mathcal{V}_M{}^{\underline{N}} = \Big(\mathcal{V}_M{}^{ij}, \mathcal{V}_{M\,kl}\Big) = \begin{pmatrix} \mathcal{V}_\Lambda{}^{ij} & \mathcal{V}_{\Lambda\,kl} \\ \mathcal{V}^{\Sigma\,ij} & \mathcal{V}^{\Sigma}{}_{kl} \end{pmatrix} . \tag{3.3}$$

This matrix thus transforms under rigid $\mathrm{E}_{7(7)}$ from the left and under local SU(8) from the right. It does not really constitute an element of $\mathrm{E}_{7(7)}$, but it is equal to a constant matrix (to account for the different bases adopted on both sides) times a space-time dependent

[4] Strictly speaking the isotropy group equals $\mathrm{SU}(8)/\mathbf{Z}_2$.

element of $E_{7(7)}$. We note the following useful properties of $\mathcal{V}_M{}^{\underline{N}}$, which also fix the normalization,

$$\begin{aligned}
\mathcal{V}_M{}^{ij}\,\mathcal{V}_{N\,ij} - \mathcal{V}_{M\,ij}\,\mathcal{V}_N{}^{ij} &= \mathrm{i}\,\Omega_{MN}\,, \\
\Omega^{MN}\,\mathcal{V}_M{}^{ij}\,\mathcal{V}_{N\,kl} &= \mathrm{i}\,\delta^{ij}{}_{kl}\,, \\
\Omega^{MN}\,\mathcal{V}_M{}^{ij}\,\mathcal{V}_N{}^{kl} &= 0\,.
\end{aligned} \tag{3.4}$$

The sign on the right-hand side is determined by the relative phase between $\mathcal{V}_\Lambda{}^{ij}$ and $\mathcal{V}^{\Lambda ij}$. Because we have already fixed the definition of the $E_{7(7)}$ transformations on the field strenghts $(F_{\mu\nu}{}^\Lambda, G_{\mu\nu\Lambda})$, we can no longer adjust this relative phase. Therefore we must distinguish two different cases characterized by the sign on the right-hand side of (3.4). As it turns out, supersymmetry selects the sign shown above.

The equations (3.3) and (3.4) imply that the inverse coset representative $\mathcal{V}^{-1}$ reads,

$$[\mathcal{V}^{-1}]_{\underline{M}}{}^N = \mathrm{i}\Omega^{NP}\Big(-\mathcal{V}_{P\,ij}, \mathcal{V}_P{}^{kl}\Big) = \begin{pmatrix} -\mathrm{i}\mathcal{V}^\Lambda{}_{ij} & \mathrm{i}\mathcal{V}_{\Sigma\,ij} \\ \mathrm{i}\mathcal{V}^{\Lambda\,kl} & -\mathrm{i}\mathcal{V}_\Sigma{}^{kl} \end{pmatrix}. \tag{3.5}$$

The most relevant restriction is, however, not captured by (3.4), namely that $\mathcal{V}_M{}^{\underline{N}}$ can be written as a constant tensor $\mathring{\mathcal{V}}_M{}^Z$ times a space-time dependent $E_{7(7)}$ matrix $\mathcal{V}_Z{}^{\underline{N}}(x)$. The latter 56×56 matrix, sometimes called the 56-bein, is usually expressed in the form,

$$\mathcal{V}_Z{}^{\underline{N}}(x) = \begin{pmatrix} u^{ij}{}_{IJ}(x) & -v_{klIJ}(x) \\ -v^{ijKL}(x) & u_{kl}{}^{KL}(x) \end{pmatrix}. \tag{3.6}$$

The indices $I, J, \ldots$ and $i, j, \ldots$ take the values $1, \ldots, 8$, so that there are 28 antisymmetrized index pairs representing the matrix indices of $\mathcal{V}$; the row indices are $Z = ([IJ], [KL])$, and the column indices are $\underline{N} = ([ij], [kl])$, so as to remain consistent with the conventions of [1]. The above matrix is pseudoreal and belongs to $E_{7(7)} \subset \mathrm{Sp}(56;\mathbb{R})$ in the fundamental representation. We use the convention where $u^{ij}{}_{IJ} = (u_{ij}{}^{IJ})^*$ and $v_{ijIJ} = (v^{ijIJ})^*$. The indices $i, j, \ldots$ refer to local SU(8) transformations and capital indices $I, J, \ldots$ are subject to rigid $E_{7(7)}$ transformations.

A crucial question regards the nature of the constant matrix $\mathring{\mathcal{V}}$. Obviously (3.4) leaves the freedom to perform a redefinition by acting with an $\mathrm{Sp}(56;\mathbb{R})$ transformation from the left. Because $\mathcal{V}_M$ is defined with a lower index, such a transformation acts as follows,

$$\begin{aligned}
\mathcal{V}_\Lambda{}^{ij} &\to V_\Lambda{}^\Sigma\,\mathcal{V}_\Sigma{}^{ij} - W_{\Lambda\Sigma}\,\mathcal{V}^{\Sigma ij}\,, \\
\mathcal{V}^{\Lambda ij} &\to U^\Lambda{}_\Sigma\,\mathcal{V}^{\Sigma ij} - Z^{\Lambda\Sigma}\,\mathcal{V}_\Sigma{}^{ij}\,.
\end{aligned} \tag{3.7}$$

These redefinitions lead to an obvious ambiguity in the definition of $\mathring{\mathcal{V}}$ and correspondingly in the definition of $\mathcal{V}_\Lambda{}^{ij}$ and $\mathcal{V}^{\Lambda ij}$. However, some of this ambiguity can be removed, either by absorbing an $E_{7(7)}$ transformation emerging on the right into the definition of $\mathcal{V}_{\underline{P}}{}^Z(x)$, or by absorbing an GL(28) transformation emerging on the left into the definition of the gauge fields. The ambiguity thus takes the form of an $E_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$ matrix (or rather its inverse) [23, 7]. The Lagrangian will implicitly depend on this matrix, as it will be written in terms of $\mathcal{V}_\Lambda{}^{ij}$ and $\mathcal{V}^{\Lambda ij}$.

Let us now briefly discuss the pseudoreal representation of $\mathrm{E}_{7(7)}$. The maximal compact subgroup SU(8) coincides with the R-symmetry group of four-dimensional $N = 8$ supersymmetry which is relevant for the fermions, as the chiral and antichiral gravitino and spinor fields transform in the $\mathbf{8} + \overline{\mathbf{8}}$, and $\mathbf{56} + \overline{\mathbf{56}}$ representation of that group. Therefore the pseudoreal basis, based on the SU(8) decomposition $\mathbf{56} \to \overline{\mathbf{28}} + \mathbf{28}$, is particularly relevant. In the $\mathbf{56}$ representation, the basis vectors in the $\mathbf{56}$ representation are then denoted by (z_{IJ}, z^{KL}) with $z^{IJ} = (z_{IJ})^*$; here the indices are antisymmetrized index pairs $[IJ]$ and $[KL]$ and $I, J, K, L = 1, \ldots, 8$. The z^{IJ} transform according to the $\mathbf{28}$ representation of SU(8). Infinitesimal $\mathrm{Sp}(56; \mathbb{R})$ transformations now take the form,

$$\begin{aligned} \delta z_{IJ} &= \Lambda_{IJ}{}^{KL}\, z_{KL} + \Sigma_{IJKL}\, z^{KL} \,, \\ \delta z^{IJ} &= \Lambda^{IJ}{}_{KL}\, z^{KL} + \Sigma^{IJKL}\, z_{KL} \,, \end{aligned} \tag{3.8}$$

where $\Lambda_{IJ}{}^{KL}$ and Σ_{IJKL} are subject to the conditions

$$(\Lambda_{IJ}{}^{KL})^* = \Lambda^{IJ}{}_{KL} = -\Lambda_{KL}{}^{IJ} \,, \qquad (\Sigma_{IJKL})^* = \Sigma^{KLIJ} \,. \tag{3.9}$$

The matrices $\Lambda_{IJ}{}^{KL}$ are associated with the maximal compact U(28) subgroup. In this basis the invariant skew-symmetric tensor Ω is proportional to (2.15). The $\mathrm{E}_{7(7)}$ subgroup of $\mathrm{Sp}(56; \mathbb{R})$ is obtained for fully antisymmetric Σ^{IJKL} with the additional restrictions,

$$\begin{aligned} &\Lambda_{IJ}{}^{KL} = \delta_{[I}{}^{[K}\, \Lambda_{J]}{}^{L]} \,, \qquad \Lambda_I{}^J = -\Lambda^J{}_I \,, \\ &\Lambda_I{}^I = 0 \,, \qquad \Sigma_{IJKL} = \frac{1}{24} \varepsilon_{IJKLMNPQ}\, \Sigma^{MNPQ} \,. \end{aligned} \tag{3.10}$$

The $\Lambda_I{}^J$ generate the group SU(8). Closure of the full algebra is ensured by the fact that two tensors Σ_1 and Σ_2 satisfy the relation

$$\begin{aligned} &\Sigma_{1\,IJMN}\, \Sigma_2{}^{MNKL} - \Sigma_{2\,IJMN}\, \Sigma_1{}^{MNKL} \\ &= \frac{2}{3}\, \delta_{[I}{}^{[K} \left(\Sigma_{1\,J]MNP}\, \Sigma_2{}^{L]MNP} - \Sigma_{2\,J]MNP}\, \Sigma_1{}^{L]MNP} \right) , \end{aligned} \tag{3.11}$$

which follows from the selfduality of Σ. All this is in accord with the branching of the adjoint representation of $\mathrm{E}_{7(7)}$ with respect to its SU(8) subgroup: $\mathbf{133} \to \mathbf{63} + \mathbf{70}$.

Before returning to the T-tensor, let us first reconsider the representation of the scalar fields based on $\mathcal{V}_\Lambda{}^{ij}$ and $\mathcal{V}^{\Lambda ij}$. Under arbitrary variations of the $\mathrm{E}_{7(7)}$ matrix (3.6) we note the result,

$$[\mathcal{V}^{-1}]_{\underline{M}}{}^N\, \delta\mathcal{V}_N{}^{\underline{P}} = [\mathcal{V}^{-1}]_{\underline{M}}{}^Z\, \delta\mathcal{V}_Z{}^{\underline{P}} \,, \tag{3.12}$$

which follows from the fact that the constant matrix $\overset{\circ}{\mathcal{V}}$ cancels in the expression on the left-hand side. This observation leads to

$$\begin{aligned} \mathcal{V}_{Mij}\, \delta\mathcal{V}_N{}^{kl}\, \Omega^{MN} &= -\mathrm{i} \Big(u_{ij}{}^{IJ}\, \delta u^{kl}{}_{IJ} - v_{ijIJ}\, \delta v^{klIJ} \Big) \,, \\ \mathcal{V}_{Mij}\, \delta\mathcal{V}_{Nkl}\, \Omega^{MN} &= -\mathrm{i} \Big(v_{ijIJ}\, \delta u_{kl}{}^{IJ} - u_{ij}{}^{IJ}\, \delta v_{klIJ} \Big) \,. \end{aligned} \tag{3.13}$$

The expression on the right-hand side shows that the equation (3.12) can be decomposed into the generators of $\mathrm{E}_{7(7)}$. The first term should be proportional to the SU(8) generators in the **28** representation, and the second term should belong to the **70** representation. Using these restrictions, we derive,

$$\begin{aligned}
\mathcal{V}_{Mij}\,\delta\mathcal{V}_N{}^{kl}\,\Omega^{MN} &= \frac{3}{2}\delta_{[i}{}^{[k}\,\mathcal{V}_{Mj]m}\,\delta\mathcal{V}_N{}^{l]m}\,\Omega^{MN}\,,\\
\mathcal{V}_{Mij}\,\delta\mathcal{V}_{Nkl}\,\Omega^{MN} &= \mathcal{V}_{M[ij}\,\delta\mathcal{V}_{Nkl]}\,\Omega^{MN}\,,\\
\mathcal{V}_M{}^{ij}\,\delta\mathcal{V}_N{}^{kl}\,\Omega^{MN} &= -\frac{1}{24}\varepsilon^{ijklmnpq}\,\mathcal{V}_{Mmn}\,\delta\mathcal{V}_{Npq}\,\Omega^{MN}\,.
\end{aligned}\tag{3.14}$$

In what follows these equations play an important role.

Let us now return to the T-tensor. First we draw attention to the fact that, when treating the embedding tensor as a spurionic object that transforms under the duality group, the equations of motion, the Bianchi identities and the transformation rules remain formally invariant under $\mathrm{E}_{7(7)}$. Under the latter $\Theta_M{}^\alpha$ would transform as $\Theta_M{}^\alpha\,t_\alpha \to g_M{}^N\,\Theta_N{}^\alpha\,(g\,t_\alpha g^{-1})$, with $g\in\mathrm{E}_{7(7)}$. The same observation applies to the T-tensor. To make this more explicit we note that *every* variation of the coset representative can be expressed as a (possibly field-dependent) $\mathrm{E}_{7(7)}$ transformation acting on $\mathcal{V}$ from the right. For example, a rigid $\mathrm{E}_{7(7)}$ transformation acting from the left can be rewritten as a field-dependent transformation from the right,

$$\mathcal{V}\to\mathcal{V}'=g\,\mathcal{V}=\mathcal{V}\,\sigma^{-1}\,,\tag{3.15}$$

with $\sigma^{-1}=\mathcal{V}^{-1}\,g\,\mathcal{V}\in\mathrm{E}_{7(7)}$, but also a supersymmetry transformation can be written in this form. Consequently, these variations of $\mathcal{V}$ induce the following transformation of the T-tensor,

$$T_{\underline{MN}}{}^{\underline{P}}\to T'_{\underline{MN}}{}^{\underline{P}}=\sigma_{\underline{M}}{}^{\underline{Q}}\,\sigma_{\underline{N}}{}^{\underline{R}}\,(\sigma^{-1})_{\underline{S}}{}^{\underline{P}}\,T_{\underline{QR}}{}^{\underline{S}}\,.\tag{3.16}$$

This implies that the T-tensor constitutes a representation of $\mathrm{E}_{7(7)}$. Observe that this is *not* an invariance statement; rather it means that the T-tensor (irrespective of the choice for the corresponding embedding tensor) varies under supersymmetry or any other transformation in a way that can be written as a (possibly field-dependent) $\mathrm{E}_{7(7)}$-transformation. Note also that the transformation assignment of the embedding tensor and the T-tensor are opposite in view of the relationship between g and σ, something that is important in practical applications.

Subsequently we determine the T-tensor according to (3.2). First we define

$$\begin{aligned}
\Omega^{NP}\,\mathcal{V}_{N\,ij}X_{MP}{}^Q\,\mathcal{V}_Q{}^{kl} &= -\,\mathrm{i}\mathcal{Q}_{M\,ij}{}^{kl}\,,\\
\Omega^{NP}\,\mathcal{V}_{N\,ij}X_{MP}{}^Q\,\mathcal{V}_{Qkl} &= -\,\mathrm{i}\mathcal{P}_{M\,ijkl}\,.
\end{aligned}\tag{3.17}$$

We note that $\mathcal{Q}_M$ and $\mathcal{P}_M$ are subject to constraints,

$$Z^{M,\alpha}\,\mathcal{Q}_{Mij}{}^{kl}=0\,,\qquad Z^{M,\alpha}\,\mathcal{P}_{Mijkl}=0\,,\tag{3.18}$$

by virtue of the quadratic constraint (2.16). The tensor $Z^{M,\alpha}$ was defined in (2.22). For the convenience of the reader, we also note the relation,

$$X_{MN}{}^P\,\mathcal{V}_P{}^{ij}=\mathcal{P}_M{}^{ijkl}\,\mathcal{V}_{Nkl}+\mathcal{Q}_{Mkl}{}^{ij}\,\mathcal{V}_N{}^{kl}\,.\tag{3.19}$$

The generators X_M define a subgroup of $\mathrm{E}_{7(7)}$ in a certain electric/magnetic duality basis, which in (3.17) is converted to the pseudoreal representation. Compatibility with the Lie algebra of $\mathrm{E}_{7(7)}$ implies that $\mathcal{P}_M{}^{ijkl}$ is a selfdual SU(8) tensor,

$$\mathcal{P}_M{}^{ijkl} = \frac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{M\,mnpq}\,, \tag{3.20}$$

and that $\mathcal{Q}_M$ transforms as a connection associated with SU(8). Hence, $\mathcal{Q}_{M\,ij}{}^{kl}$ satisfies the decomposition,

$$\mathcal{Q}_{M\,ij}{}^{kl} = \delta_{[i}{}^{[k}\,\mathcal{Q}_{M\,j]}{}^{l]}\,, \tag{3.21}$$

with $\mathcal{Q}_M{}^i{}_j = -\mathcal{Q}_{Mj}{}^i$ and $\mathcal{Q}_{Mi}{}^i = 0$. Decomposing

$$T_{\underline{MN}}{}^{\underline{P}} = \left(T_{ij\underline{N}}{}^{\underline{P}}, T^{kl}{}_{\underline{N}}{}^{\underline{P}}\right)\,, \tag{3.22}$$

we write the components of the T-tensor in matrix notation,

$$T_{ij} = \begin{pmatrix} -\frac{2}{3}\delta_{[k}{}^{[p}\,T^{q]}{}_{l]ij} & \frac{1}{24}\varepsilon_{klrstuvw}\,T^{tuvw}{}_{ij} \\ & \\ T^{mnpq}{}_{ij} & \frac{2}{3}\delta_{[r}{}^{[m}\,T^{n]}{}_{s]ij} \end{pmatrix}\,, \tag{3.23}$$

where $({}_{[kl]}, {}^{[mn]})$ are the row indices and $({}^{[pq]}, {}_{[rs]})$ the column indices, and

$$T^{ij} = \begin{pmatrix} \frac{2}{3}\delta_{[k}{}^{[p}\,T_{l]}{}^{q]ij} & T_{klrs}{}^{ij} \\ & \\ \frac{1}{24}\varepsilon^{mnpqtuvw}T_{tuvw}{}^{ij} & -\frac{2}{3}\delta_{[r}{}^{[m}\,T_{s]}{}^{n]ij} \end{pmatrix}\,. \tag{3.24}$$

Multiplicative factors have been included to make contact with the definitions of [1, 23, 7]. In order to belong to the Lie algebra of $\mathrm{E}_{7(7)}$, the matrix blocks in the above expressions satisfy $T_k{}^{kij} = 0$ and $T^{klmn}{}_{ij} = T^{[klmn]}{}_{ij}$. Note that we always use the convention where complex conjugation is effected by raising and lowering of indices SU(8).

Comparing the above expressions, one can directly establish the following expressions,[5]

$$\begin{aligned} T_k{}^{lij} &= \frac{3}{4}\mathrm{i}\,\Omega^{MN}\,\mathcal{Q}_{M\,k}{}^{l}\,\mathcal{V}_N{}^{ij}\,, \\ T_{klmn}{}^{ij} &= \frac{1}{2}\mathrm{i}\,\Omega^{MN}\,\mathcal{P}_{M\,klmn}\,\mathcal{V}_N{}^{ij}\,. \end{aligned} \tag{3.25}$$

Note that so far no constraints have been imposed on the T-tensor.

We already noted that every variation of the coset representative can be cast in the form of an $\mathrm{E}_{7(7)}$ transformation acting on the right of $\mathcal{V}$. This implies that any variation of the T-tensor is again proportional to the T-tensor itself (c.f. (3.16)). In view of the covariance under the SU(8) subgroup, the only relevant variation is therefore

$$\mathcal{V} \to \mathcal{V}\begin{pmatrix} 0 & \overline{\Sigma} \\ \Sigma & 0 \end{pmatrix}\,. \tag{3.26}$$

[5]Unlike in the original definition (3.2) the $\mathcal{V}_M$ are only proportional to an $\mathrm{E}_{7(7)}$ group element, so that the proportionality factor in (3.25) is not intrinsically defined. Our choice for this factor is such that our results remain as closely related as possible to the original expressions of [1].

In this way one can derive,

$$\begin{aligned}
\delta T_i{}^{jkl} &= \Sigma^{jmnp}\, T_{imnp}{}^{kl} - \frac{1}{24}\varepsilon^{jmnpqrst}\, \Sigma_{imnp}\, T_{qrst}{}^{kl} + \Sigma^{klmn}\, T^j{}_{imn} \\
&= 2\,\Sigma^{jmnp}\, T_{imnp}{}^{kl} - \frac{1}{4}\,\delta_i{}^j\, \Sigma^{mnpq}\, T_{mnpq}{}^{kl} + \Sigma^{klmn}\, T^j{}_{imn}\,, \\
\delta T_{ijkl}{}^{mn} &= -\frac{4}{3}\Sigma_{p[ijk}\, T_{l]}{}^{pmn} - \frac{1}{24}\varepsilon_{ijklpqrs}\, \Sigma^{mntu}\, T^{pqrs}{}_{tu}\,.
\end{aligned} \tag{3.27}$$

This formula can be used for evaluating, for instance, space-time derivatives or supersymmetry variations of the T-tensor, where one must choose the appropriate expressions for $\Sigma, \overline{\Sigma} \propto \mathcal{V}^{-1}\,\delta\mathcal{V}$.

Armed with these results we can now proceed and derive the constraints on the T-tensor induced by the embedding tensor constraints discussed in the previous section. First of all, as a consequence of (2.12), the T-tensor is constrained to the **912** representation of $\mathrm{E}_{7(7)}$, which decomposes into a **36** and a **420** representation of SU(8). This shows that there must be a proportionality relation between $T^{klmn}{}_{ij}$ and $\delta_{[i}{}^{[k}\,T_{j]}{}^{lmn]}$, as both sides can only contain the **420** representation. Checking the consistency of this with (3.27), it follows that

$$\begin{aligned}
T^{klmn}{}_{ij} &= -\frac{4}{3}\delta_{[i}{}^{[k}\, T_{j]}{}^{lmn]}\,, \\
T_i{}^{jkl} &= -\frac{3}{4}\, A_{2i}{}^{jkl} - \frac{3}{2} A_1{}^{j[k}\,\delta^{l]}{}_i\,,
\end{aligned} \tag{3.28}$$

where $A_{2i}{}^{jkl} = A_{2i}{}^{[jkl]}$, $A_{2i}{}^{jki} = 0$ and $A_1^{[ij]} = 0$, so that $T_i{}^{[ijk]} = 0$. Clearly A_1 and A_2 represent the **36** and **420** representations of SU(8), respectively. These results are not new and were first given in [1], but we prefer to give a self-contained derivation here to demonstrate how to cast the group-theoretical restrictions into the equations that one needs for the Lagrangian. The SU(8) tensors A_1 and A_2 appear in the Lagrangian in the masslike terms and in the scalar potential that we will present in the next section. In fact, the supersymmetry of the action to first order of the gauge coupling constant g, depends crucially on (3.28). Note that none of these results depend on the actual gauge group. The only requirement is that the embedding tensor satisfies the constraints discussed in the previous section.

We now turn to a discussion of the constraints that are quadratic in the T-tensor. These constraints are sufficient for proving the supersymmetry of the action to second order in g. In section 2 we presented two alternative expressions for the quadratic constraint. One is (2.16), which can be rewritten as an equation for the T-tensor after suitable multiplication with $\mathcal{V}$. The results, which coincide with the ones derived in [1, 7], take the form,

$$\begin{aligned}
T^k{}_{lij}\, T_n{}^{mij} - T_l^{kij}\, T^m{}_{nij} &= 0\,, \\
T^k{}_{lij}\, T_{mnpq}{}^{ij} + \frac{1}{24}\varepsilon_{mnpqrstu}\, T_l^{kij}\, T^{rstu}{}_{ij} &= 0\,, \\
T_{irst}{}^{vw}\, T^{jrst}{}_{vw} - \frac{1}{8}\delta_i^j\, T_{rstu}{}^{vw}\, T^{rstu}{}_{vw} &= 0\,, \\
T_{ijkr}{}^{vw}\, T^{mnpr}{}_{vw} - \frac{9}{4}\delta_{[i}^{[m}\, T_{jk]rs}{}^{vw}\, T^{np]rs}{}_{vw} + \frac{1}{16}\delta^{mnp}_{i\,j\,k}\, T_{rstu}{}^{vw}\, T^{rstu}{}_{vw} &= 0\,,
\end{aligned} \tag{3.29}$$

where in the last identity the antisymmetrization does not include the indices v, w. Substituting the results of (3.28), these equations reduce to,

$$
\begin{aligned}
&A_2{}^k{}_{lij}\, A_{2n}{}^{mij} - A_{2l}{}^{kij}\, A_2{}^m{}_{nij} - 4 A_2{}^{(k}{}_{lni} A_1^{m)i} - 4 A_{2(n}{}^{mki} A_{1l)i} \\
&\qquad\qquad\qquad - 2\delta_l^m A_{1ni} A_1{}^{ki} + 2\delta_n^k A_{1li} A_1{}^{mi} = 0\,, \\
&A_2{}^i{}_{jk[m}\, A_2{}^k{}_{npq]} + A_{1jk}\delta^i_{[m} A_2{}^k{}_{npq]} - A_{1j[m} A_2{}^i{}_{npq]} \\
&\qquad + \frac{1}{24}\varepsilon_{mnpqrstu}\, (A_{2j}{}^{ikr}\, A_{2k}{}^{stu} + A_1^{jk}\delta^r_j A_{2k}{}^{stu} - A_1^{ir} A_{2j}{}^{stu}) = 0\,, \\
&\qquad 9\, A_2{}^m{}_{ikl}\, A_{2m}{}^{jkl} - A_2{}^j{}_{klm}\, A_{2i}{}^{klm} - \delta_i{}^j\, A_2{}^n{}_{klm}\, A_{2n}{}^{klm} = 0\,, \\
&A_2{}^r{}_{ijk}\, A_{2r}{}^{mnp} - 9\, A_2{}^{[m}{}_{r[ij}\, A_{2k]}{}^{np]r} - 9\, \delta_{[i}{}^{[m}\, A_2{}^n{}_{rsj}\, A_{2k]}{}^{p]rs} \\
&\qquad\qquad -9\, \delta_{[i\,j}{}^{[mn}\, A_2{}^u{}_{k]rs}\, A_{2u}{}^{p]rs} + \delta^{mnp}_{i\,j\,k}\, A_2{}^u{}_{rst}\, A_{2u}{}^{rst} = 0\,,
\end{aligned}
\tag{3.30}
$$

where the antisymmetrizations in the last equation apply to the index triples $[ijk]$ and $[mnp]$. Note that the representation content of these four constraint equations is $\mathbf{945} + \overline{\mathbf{945}} + \mathbf{63}$, $\mathbf{3584} + \mathbf{378} + \overline{\mathbf{378}} + \mathbf{70}$, $\mathbf{63}$ and $\mathbf{2352}$, respectively.

As we intend to demonstrate in the following, consistent gaugings are characterized by embedding tensors that satisfy two constraints (2.12) and (2.16), one linear and one quadratic in this tensor. These two constraints lead to corresponding constraints on the T-tensor, namely (3.28) and (3.30).

4. The Lagrangian and transformation rules

In principle the Lagrangian and transformation rules are known from [1], but we have to convert to the unconventional definition of the coset representative. Furthermore we have to make contact with the formalism of [16] to incorporate possible magnetic charges. The reader who wishes to avoid the complications associated with the magnetic charges, can simply assume that an appropriate electric/magnetic duality transformation has been performed so that there are only electric charges (implying that $\Theta^{\Lambda\alpha} = 0$). But as we have indicated previously, there is a variety of reasons why it is advantageous to remain in a more general electric/magnetic duality frame.

4.1 Coset geometry

The first issue that we have to address is related to the coset representative of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$. In particular we have to write the composite SU(8) gauge fields $\mathcal{Q}_\mu$ and the tensor $\mathcal{P}_\mu$ appearing in the kinetic term for the scalar fields in terms of the $\mathcal{V}_M{}^{ij}$. This proceeds in the standard way. We assume the presence of 56 gauge fields $A_\mu{}^M$ which couple to the charges X_M as in (2.5). The covariant derivative,

$$
\mathcal{D}_\mu \mathcal{V}_M{}^{ij} = \partial_\mu \mathcal{V}_M{}^{ij} - \mathcal{Q}_{\mu\, kl}{}^{ij}\, \mathcal{V}_M{}^{kl} - g\, A_\mu{}^P X_{PM}{}^N\, \mathcal{V}_N{}^{ij}\,, \tag{4.1}
$$

is covariant with respect to SU(8), with corresponding connection

$$
\mathcal{Q}_{\mu\, ij}{}^{kl} = \delta_{[i}{}^{[k}\, \mathcal{Q}_{\mu\, j]}{}^{l]}\,, \tag{4.2}
$$

with $\mathcal{Q}_{\mu}{}^{i}{}_{j} = -\mathcal{Q}_{\mu j}{}^{i}$ and $\mathcal{Q}_{\mu i}{}^{i} = 0$. Furthermore it is covariant under the optional gauge transformations with generators X_M and connections $A_\mu{}^M$. The SU(8) connection is, however, not an independent field and determined by the condition,

$$\Omega^{MN}\,\mathcal{V}_{Mij}\,\mathcal{D}_\mu\mathcal{V}_N{}^{kl} = 0\,, \tag{4.3}$$

which yields

$$\mathcal{Q}_{\mu\,i}{}^{j} = \frac{2}{3}\mathrm{i}(\mathcal{V}_{\Lambda\,ik}\,\partial_\mu\mathcal{V}^{\Lambda\,jk} - \mathcal{V}^{\Lambda}{}_{ik}\,\partial_\mu\mathcal{V}_\Lambda{}^{jk}) - g\,A_\mu{}^M\,\mathcal{Q}_{M\,i}{}^{j}\,, \tag{4.4}$$

where $\mathcal{Q}_{M\,i}{}^{j}$ is defined by (3.17).

In addition we define an SU(8) tensor $\mathcal{P}_{\mu\,ijkl}$ which is invariant under the optional gauge group G_g,

$$\mathcal{P}_{\mu\,ijkl} = \mathrm{i}\Omega^{MN}\,\mathcal{V}_{Mij}\,\mathcal{D}_\mu\mathcal{V}_{Nkl} = \mathrm{i}(\mathcal{V}_{\Lambda\,ij}\,\mathcal{D}_\mu\mathcal{V}^{\Lambda}{}_{kl} - \mathcal{V}^{\Lambda}{}_{ij}\,\mathcal{D}_\mu\mathcal{V}_{\Lambda\,kl})\,, \tag{4.5}$$

where the gauge fields contribute through the covariant derivative, leading to $-g\,A_\mu{}^M\,\mathcal{P}_{M\,ijkl}$. Compatibility with the Lie algebra of $\mathrm{E}_{7(7)}$ implies that $\mathcal{P}_{\mu\,ijkl}$ is a selfdual SU(8) tensor,

$$\mathcal{P}_\mu{}^{ijkl} = \frac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{\mu\,mnpq}\,. \tag{4.6}$$

Furthermore we note the useful identity,

$$\mathcal{D}_\mu\mathcal{V}_M{}^{ij} = \mathcal{P}_\mu{}^{ijkl}\,\mathcal{V}_{Mkl}\,. \tag{4.7}$$

Applying a second derivative to (4.3) (4.7) leads to integrability conditions known as the Cartan-Maurer equations,

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q})_i{}^j &= -\frac{4}{3}\,\mathcal{P}_{[\mu}{}^{jklm}\,\mathcal{P}_{\nu]iklm} - g\,\mathcal{F}_{\mu\nu}{}^M\,\mathcal{Q}_{M\,i}{}^j\,,\\ D_{[\mu}\mathcal{P}_{\nu]}{}^{ijkl} &= -\frac{1}{2}g\,\mathcal{F}_{\mu\nu}{}^M\,\mathcal{P}_M{}^{ijkl}\,, \end{aligned} \tag{4.8}$$

where $\mathcal{Q}_{M\,i}{}^j$ and $\mathcal{P}_M{}^{ijkl}$ are defined by (3.17),

$$F(\mathcal{Q})_{\mu\nu i}{}^j = \partial_\mu\mathcal{Q}_{\nu i}{}^j - \partial_\nu\mathcal{Q}_{\mu i}{}^j + \mathcal{Q}_{[\mu i}{}^k\,\mathcal{Q}_{\nu]k}{}^j\,, \tag{4.9}$$

is the SU(8) field strength, and $\mathcal{F}_{\mu\nu}{}^M$ was already defined in (2.48). These Cartan-Maurer equations are important for deriving the supersymmetry of the action. The order-g terms violate the supersymmetry of the original ungauged Lagrangian as they induce new supersymmetry variations of the gravitino kinetic terms and the Noether term, which are proportional to the field strengths $\mathcal{F}_{\mu\nu}{}^M$ and also to the T-tensor.

4.2 The ungauged Lagrangian

In this subsection we briefly introduce the ungauged Lagrangian of $N = 8$ supergravity in the notation of this paper. Up to terms proportional to the field equations of the gauge fields, this Lagrangian is invariant under an $\mathrm{E}_{7(7)}$ subgroup of the $\mathrm{Sp}(56,\mathbb{R})$ electric/magnetic duality group. The most crucial part of the Lagrangian concerns the 28

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electric vector fields $A_\mu{}^\Lambda$ (their magnetic duals $A_{\mu\Lambda}$ are absent as we already discussed in subsection 2.2), which are only invariant under a subgroup of $\mathrm{E}_{7(7)}$. The field equations for these vector fields and the Bianchi identities for their field strengths constitute 56 equations, given in (2.32), which are subject to electric/magnetic duality transformations. Only the vector field strengths $F_{\mu\nu}{}^M$ and the scalar fields contained in $\mathcal{V}_M{}^{ij}$ are subject to the $\mathrm{E}_{7(7)}$ transformations.

The generic gauge field Lagrangian, parametrized as in [16], was given in (2.36) and contains moment couplings of the field strength $F_{\mu\nu}{}^\Lambda$ with an operator $\mathcal{O}_{\mu\nu\Lambda}$ which is quadratic in the fermions. Here we will discuss the explict form of $\mathcal{O}_{\mu\nu\Lambda}$ and of $\mathcal{N}_{\Lambda\Sigma}$. We start from the 56 field strengths $G_{\mu\nu}{}^M$, introduced in susbsection 2.2, which transform under the $\mathrm{E}_{7(7)}$ transformations, which are embedded in the $\mathrm{Sp}(56,\mathbb{R})$ electric/magnetic duality group. From these field strengths and $\mathcal{V}_M{}^{ij}$ and its complex conjugate, we can construct $\mathrm{E}_{7(7)}$ invariant tensors. Specifically, consider the 56 $\mathrm{E}_{7(7)}$ invariant tensors, $\mathcal{V}_M{}^{ij}G^+_{\mu\nu}{}^M$ and $\mathcal{V}_{M\,ij}G^+_{\mu\nu}{}^M$, and their anti-selfdual counterparts that follow by hermitean conjugation. The fermionic bilinears $\mathcal{O}_{\mu\nu\Lambda}$ are proportional to the following SU(8) covariant expression [24, 25, 18],

$$\mathcal{O}^+_{\mu\nu}{}^{ij} = \frac{1}{2}\sqrt{2}\,\bar\psi^i_\rho\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]}\psi^j_\sigma - \frac{1}{2}\bar\psi_{\rho\,k}\gamma_{\mu\nu}\gamma^\rho\chi^{ijk} - \frac{1}{144}\sqrt{2}\,\varepsilon^{ijklmnpq}\bar\chi_{klm}\gamma_{\mu\nu}\chi_{npq}\,, \tag{4.10}$$

which is selfdual and transforming in the $\overline{\mathbf{28}}$ representation of SU(8). Its complex conjugate is anti-selfdual and transforms in the **28** representation. The fact that only a single tensor of fermionic bilinears appears in the relation between the field strengths $F_{\mu\nu}{}^\Lambda$ and the dual field strengths $G_{\mu\nu\Lambda}$, implies that this relation must coincide with the following $\mathrm{E}_{7(7)}$ invariant equation,[6]

$$\mathcal{V}_M{}^{ij}\,G^+_{\mu\nu}{}^M = -\frac{1}{2}\mathcal{O}^+_{\mu\nu}{}^{ij}\,. \tag{4.11}$$

The independent combination, $\mathcal{V}_{M\,ij}G^+_{\mu\nu}{}^M$, defines an SU(8) covariant tensor,

$$F^+_{\mu\nu\,ij} \equiv \mathcal{V}_{M\,ij}\,G^+_{\mu\nu}{}^M\,, \tag{4.12}$$

which will appear in the supersymmetry transformations of the fermions. In this way both the $\mathrm{E}_{7(7)}$ invariance and the SU(8) covariance of the supersymmetry transformations will be ensured. Using (3.4), we derive the following equation,

$$G^+_{\mu\nu}{}^M = \mathrm{i}\Omega^{MN}\left[\mathcal{V}_N{}^{ij}F^+_{\mu\nu\,ij} + \frac{1}{2}\mathcal{V}_{N\,ij}\mathcal{O}^+_{\mu\nu}{}^{ij}\right]. \tag{4.13}$$

Furthermore, comparison of (4.11) to (2.37) leads to a determination of $\mathcal{N}_{\Lambda\Sigma}$ and $\mathcal{O}^+_{\mu\nu\Lambda}$,

$$\begin{aligned}\mathcal{V}^{\Sigma ij}\,\mathcal{N}_{\Lambda\Sigma} &= -\mathcal{V}_\Lambda{}^{ij}\,,\\ \mathcal{V}^{\Lambda ij}\,\mathcal{O}^+_{\mu\nu\Lambda} &= \frac{1}{4}\mathrm{i}\,\mathcal{O}^+_{\mu\nu}{}^{ij}\,.\end{aligned} \tag{4.14}$$

[6] We follow the argumentation presented in [1]. The proportionality factor on the right-hand side of the equation follows from supersymmetry.

These equations hold in any electric/magnetic duality frame and the reader may verify that (2.38) is indeed consistent with (3.7). Furthermore we note the relation,

$$[(\mathcal{N}-\bar{\mathcal{N}})^{-1}]^{\Lambda\Sigma} = \mathrm{i}\, \mathcal{V}^{\Lambda}{}_{ij}\, \mathcal{V}^{\Sigma\, ij}\,. \tag{4.15}$$

Observe that the imaginary part of the matrix $\mathcal{N}_{IJ}$ is negative so that the kinetic term in (2.36) carries the correct sign. The sign in (4.15) depends crucially on the sign adopted in (3.4). We also note the following relation

$$F^{+\,\Lambda}_{\mu\nu}\, \mathcal{O}^{+\mu\nu}_{\Lambda} = -\frac{1}{4} F^{+}_{\mu\nu\, ij}\, \mathcal{O}^{+\mu\nu\, ij} + 2\, \mathcal{O}^{+}_{\mu\nu\Lambda}\, \mathcal{O}^{+\mu\nu}_{\Sigma}\, \mathcal{V}^{\Lambda}{}_{ij}\, \mathcal{V}^{\Sigma\, ij}\,. \tag{4.16}$$

Most of the transformation rules and the Lagrangian can be deduced from [1]. As the reader may verify, they are consistent with $\mathrm{E}_{7(7)}$ and SU(8) covariance. The transformation rules can be written as follows,

$$\begin{aligned}
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i + \frac{1}{4}\sqrt{2}\,\hat{F}^{-\,ij}_{\rho\sigma}\,\gamma^{\rho\sigma}\gamma_\mu\epsilon_j + \frac{1}{4}\bar{\chi}^{ikl}\gamma^a\chi_{jkl}\,\gamma_a\gamma_\mu\epsilon^j \\
&\quad + \frac{1}{2}\sqrt{2}\,\bar{\psi}_{\mu k}\gamma^a\chi^{ijk}\,\gamma_a\epsilon_j - \frac{1}{576}\,\varepsilon^{ijklmnpq}\bar{\chi}_{klm}\gamma^{ab}\chi_{npq}\,\gamma_\mu\gamma_{ab}\epsilon_j\,, \\
\delta\chi^{ijk} &= -2\sqrt{2}\,\hat{\mathcal{P}}^{ijkl}_\mu\,\gamma^\mu\epsilon_l + \frac{3}{2}\,\hat{F}^{-\,[ij}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{k]} - \frac{1}{24}\sqrt{2}\,\varepsilon^{ijklmnpq}\bar{\chi}_{lmn}\chi_{pqr}\,\epsilon^r\,, \\
\delta e_\mu{}^a &= \bar{\epsilon}^i\gamma^a\psi_{\mu i} \;+\; \bar{\epsilon}_i\gamma^a\psi_\mu{}^i\,, \\
\delta\mathcal{V}_M{}^{ij} &= 2\sqrt{2}\,\mathcal{V}_{Mkl}\left(\bar{\epsilon}^{[i}\chi^{jkl]} + \frac{1}{24}\varepsilon^{ijklmnpq}\,\bar{\epsilon}_m\chi_{npq}\right), \\
\delta A_\mu{}^M &= -\mathrm{i}\,\Omega^{MN}\mathcal{V}_N{}^{ij}\left(\bar{\epsilon}^k\,\gamma_\mu\,\chi_{ijk} + 2\sqrt{2}\,\bar{\epsilon}_i\,\psi_{\mu j}\right) \;+\; \text{h.c.}\,.
\end{aligned} \tag{4.17}$$

Here and henceforth the caret indicates that the corresponding quantity is covariantized with respect to supersymmetry. For completeness we record the expressions for $\hat{\mathcal{P}}_\mu{}^{ijkl}$ and $\hat{F}^{+}_{\mu\nu ij}$ below,

$$\begin{aligned}
\hat{\mathcal{P}}_\mu{}^{ijkl} &= \mathcal{P}_\mu{}^{ijkl} - \sqrt{2}\left(\bar{\psi}^{[i}_\mu\,\chi^{jkl]} + \frac{1}{24}\,\varepsilon^{ijklmnpq}\,\bar{\psi}_{\mu m}\,\chi_{npq}\right), \\
\hat{F}^{+}_{\mu\nu\, ij} &= F^{+}_{\mu\nu\, ij} + \frac{1}{4}\,\bar{\psi}_\rho{}^k\gamma^\rho\gamma_{\mu\nu}\chi_{ijk} - \frac{1}{8}\sqrt{2}\,\bar{\psi}_{\rho\, i}\{\gamma_{\mu\nu},\gamma^{\rho\sigma}\}\psi_{\sigma j}\,.
\end{aligned} \tag{4.18}$$

The supercovariantized field strenghts $\hat{G}_{\mu\nu}{}^M$ then follow from (4.13) by substituting the second expression on the right-hand side,

$$\hat{G}^{+\,M}_{\mu\nu} = \mathrm{i}\Omega^{MN}\left[\mathcal{V}_N{}^{ij}\hat{F}^{+}_{\mu\nu\, ij} - \frac{1}{288}\sqrt{2}\,\mathcal{V}_{N\, ij}\varepsilon^{ijklmnpq}\,\bar{\chi}_{klm}\gamma_{\mu\nu}\chi_{npq}\right]. \tag{4.19}$$

The derivatives $\mathcal{D}_\mu$ are covariant with respect to Lorentz transformations and SU(8). For instance, we note,

$$\mathcal{D}_\mu\epsilon^i = \partial_\mu\epsilon^i - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon^i + \frac{1}{2}\mathcal{Q}_\mu{}^i{}_j\epsilon^j\,. \tag{4.20}$$

The spin connection field $\omega_\mu{}^{ab}$ is consistent with the expression one would obtain in first-order formalism, and corresponds to the following value for the torsion tensor,

$$\mathcal{D}_\mu e_\nu{}^a - \mathcal{D}_\nu e_\mu{}^a = \bar\psi_{[\mu}{}^i \gamma^a \psi_{\nu]i} + \frac{1}{12}\,\varepsilon_{\mu\nu}{}^{ab}\,\bar\chi^{ijk}\gamma_b\chi_{ijk}\,. \tag{4.21}$$

Note that we wrote down transformation rules for both electric and magnetic gauge fields $A_\mu{}^M$. However, the (ungauged) Lagrangian that we are about to introduce below does not depend on the magnetic gauge fields $A_{\mu\Lambda}$. In view of what will happen when a gauging is introduced, we will resolve this by assuming that the Lagrangian is simply invariant under an additional local gauge symmetry which acts exclusively on the magnetic gauge fields according to $\delta A_{\mu\Lambda} = \Xi_{\mu\Lambda}$, where the $\Xi_{\mu\Lambda}$ are independent space-time dependent functions. At this stage this may sound somewhat trivial, but the relevance of this approach will become clear shortly when switching on general gaugings.

The above transformations (4.17) close under commutation. In particular the commutator of two consecutive supersymmetry transformations $\delta(\epsilon_1)$ and $\delta(\epsilon_2)$ leads to the following bosonic symmetry variations,

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = \xi^\mu \hat D_\mu + \delta_{\rm L}(\epsilon^{ab}) + \delta_{\rm susy}(\epsilon_3) + \delta_{\rm SU(8)}(\Lambda^i{}_j) + \delta_{\rm gauge}(\Lambda^M) + \delta_{\rm shift}(\Xi)\,. \tag{4.22}$$

The first term indicates a general coordinate transformation, with parameter ξ^μ given by

$$\xi^\mu = 2(\bar\epsilon_2{}^i\gamma^\mu\epsilon_{1i} + \bar\epsilon_{2i}\gamma^\mu\epsilon_1{}^i)\,, \tag{4.23}$$

whose covariantized form is generated on the matter fields by a supercovariant derivative. The supersymmetry transformation parameter ϵ_3 is equal to

$$\epsilon_{3i} = -\sqrt{2}\,(\bar\epsilon_2{}^j\epsilon_1{}^k)\,\chi_{ijk}\,. \tag{4.24}$$

The gauge transformation on the abelian gauge fields is expressed in terms of the parameter,

$$\Lambda^M = -4{\rm i}\sqrt{2}\,\Omega^{MN}\,(\mathcal{V}_N{}^{ij}\,\bar\epsilon_{2i}\epsilon_{1j} - \mathcal{V}_{Nij}\,\bar\epsilon_2{}^i\epsilon_1{}^j)\,, \tag{4.25}$$

which contributes to both electric and magnetic gauge fields. For these fields the general coordinate transformation appears in the form $-\xi^\nu\,G_{\mu\nu}{}^M$. For the electric gauge field the $G_{\mu\nu}{}^\Lambda$ represents the standard field strength and this term can be written as the linear combination of a general coordinate transformation accompanied by a field-dependent gauge transformation. For the magnetic gauge fields one can take the same point of view, assuming that $G_{\mu\nu\Lambda}$ is actually the curl of $A_{\mu\Lambda}$, which is a priori possible because the equations of motion (c.f. (2.32)) imply that $G_{\mu\nu\Lambda}$ is subject to a Bianchi identity. However, one does not have to take this point of view, as the shift transformation in (4.22), which acts exclusively on the magnetic gauge fields, can always accomodate any terms that arise in the supersymmetry commutator on $A_{\mu\Lambda}$.

We refrain from quoting any results for the parameters of the Lorentz and the SU(8) transformations, as they will not play an important role in what follows. In subsection 4.4 we return to the same supersymmetry commutator in the presence of electric and magnetic charges and work out some of the results in more detail.

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The full Lagrangian for the ungauged theory can be written as follows,

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}e\,R - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\,(\bar\psi_\mu{}^i\gamma_\nu D_\rho\psi_{\sigma\,i} - \bar\psi_\mu{}^i\overleftarrow{D}_\rho\gamma_\nu\psi_{\sigma\,i}) \\
& -\frac{1}{4}\,\mathrm{i}e\Big\{\mathcal{N}_{\Lambda\Sigma}\,F^{+\,\Lambda}_{\mu\nu}\,F^{+\mu\nu\Sigma} - \bar{\mathcal{N}}_{\Lambda\Sigma}\,F^{-\,\Lambda}_{\mu\nu}\,F^{-\mu\nu\Sigma}\Big\} \\
& -\frac{1}{12}e\,(\bar\chi^{ijk}\gamma^\mu D_\mu\chi_{ijk} - \bar\chi^{ijk}\overleftarrow{D}_\mu\gamma^\mu\chi_{ijk}) - \frac{1}{12}e\,|\mathcal{P}_\mu^{ijkl}|^2 \\
& -\frac{1}{12}\sqrt{2}\,e\,\Big\{\bar\chi_{ijk}\gamma^\nu\gamma^\mu\psi_{\nu\,l}\,(\mathcal{P}_\mu^{ijkl} + \hat{\mathcal{P}}_\mu^{ijkl}) + \mathrm{h.c.}\Big\} \\
& + eF^{+\Lambda}_{\mu\nu}\,\mathcal{O}^{+\mu\nu}_\Lambda + eF^{-\Lambda}_{\mu\nu}\,\mathcal{O}^{-\mu\nu}_\Lambda - e\mathcal{V}^\Lambda{}_{ij}\,\mathcal{V}^{\Sigma ij}\,\Big[\mathcal{O}^+_{\mu\nu\Lambda}\mathcal{O}^{+\mu\nu}_\Sigma + \mathcal{O}^-_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}_\Sigma\Big] \\
& + \mathcal{L}_4\,,
\end{aligned} \tag{4.26}$$

where $\mathcal{L}_4$ contains the following SU(8) invariant terms quartic in the fermion fields,

$$\begin{aligned}
\mathcal{L}_4 = & -\frac{1}{2}e\bar\psi_\mu{}^{[i}\psi_\nu{}^{j]}\,\bar\psi^\mu{}_i\psi^\nu{}_j \\
& + \frac{1}{8}e\Big[\bar\psi_\rho{}^k\gamma^{\mu\nu}\gamma^\rho\chi_{ijk}\,(\sqrt{2}\,\bar\psi_\mu{}^i\psi_\nu{}^j + \frac{1}{2}\bar\psi_{\mu l}\gamma_\nu\chi^{ijl}) \;+\; \mathrm{h.c.}\Big] \\
& + \frac{1}{288}e\Big[\varepsilon_{ijklmnpq}\bar\chi^{ijk}\gamma^{\mu\nu}\chi^{lmn}\,(\bar\psi_\mu{}^p\psi_\nu{}^q + \frac{1}{6}\sqrt{2}\bar\psi_{\mu r}\gamma_\nu\chi^{pqr}) \;+\; \mathrm{h.c.}\Big] \\
& + \frac{1}{32}e\bar\chi^{ikl}\gamma^\mu\chi_{jkl}\,\bar\chi_{imn}\gamma_\mu\chi^{jmn} - \frac{1}{96}(\bar\chi^{ijk}\gamma^\mu\chi_{ijk})^2\,.
\end{aligned} \tag{4.27}$$

The terms of higher order in the fermions were taken from [1], where their correctness was established in the presence of the SO(8) gauging. However, in the corresponding calculations only the generic properties of the T-tensor were used, which do not depend on the choice of the gauge group. Hence these four-fermion terms must be universal. Observe that the above Lagrangian applies to any electric/magnetic duality frame because we can simply redefine the fields $\mathcal{V}_M{}^{ij}$ by an $\mathrm{Sp}(56,\mathbb{R})$ matrix.

4.3 Introducing electric and magnetic charges

Charges X_M that couple to the gauge fields $A_\mu{}^M$ are introduced in the standard way by extending covariant derivatives according to (2.5). In principle we include both electric and magnetic charges and therefore we need both electric gauge fields $A_\mu{}^\Lambda$ and magnetic gauge fields $A_{\mu\Lambda}$. The fact that the latter did not appear so far in the Lagrangian (4.26) will not immediately pose a problem, but a gauging usually induces a breaking of supersymmetry. Most of the covariant derivatives do not lead to new terms when establishing supersymmmetry, but there are variations involving the commutator of the covariant derivatives that, in the presence of the gauging, lead to the (nonabelian) field strengths $\mathcal{F}_{\mu\nu}{}^M$ defined in (2.48). These terms, which are proportional to the gauge coupling constant g, are easy to identify, as they originate exclusively from the fermion kinetic terms. They are induced

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the Cartan-Maurer equations (4.8) whose right-hand sides exhibit the extra terms proportional to the gauge coupling constant g. Collecting these terms leads to the following new variations,

$$\delta\mathcal{L} = -e\,g\,\mathcal{H}_{\mu\nu}{}^{M}\Big[\frac{1}{4}\mathcal{Q}_{Mi}{}^{j}(\bar{\epsilon}^{i}\gamma^{\rho}\gamma^{\mu\nu}\psi_{\rho j} - \bar{\epsilon}_{j}\gamma^{\mu\nu}\gamma^{\rho}\psi_{\rho}{}^{i}) \\ + \frac{1}{144}\sqrt{2}\,\mathcal{P}_{Mijkl}\,\varepsilon^{ijklmnpq}\,\bar{\chi}_{mnp}\gamma^{\mu\nu}\epsilon_{q}\Big] + \text{ h.c.}\,, \qquad (4.28)$$

where $\mathcal{Q}_{Mi}{}^{j}$ and $\mathcal{P}_{Mijkl}$ were defined in (3.17) and the replacement of $\mathcal{F}_{\mu\nu}{}^{M}$ by $\mathcal{H}_{\mu\nu}{}^{M}$ is based on (3.18).

It is, in principle, well known how these variations can be cancelled [1]. Namely, one introduces masslike terms and new supersymmetry variations for the fermions. These modifications generate (among other terms) precisely the type of variations that may cancel (4.28). The masslike terms are written as follows,

$$\mathcal{L}_{\rm masslike} = e\,g\left\{\frac{1}{2}\sqrt{2}\,A_{1\,ij}\,\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu}\psi_{\nu}{}^{j} + \frac{1}{6}A_{2i}{}^{jkl}\,\bar{\psi}_{\mu}{}^{i}\gamma^{\mu}\chi_{jkl} + A_{3}^{ijk,lmn}\,\bar{\chi}_{ijk}\,\chi_{lmn}\right\} \\ + \text{ h.c.}\,, \qquad (4.29)$$

where

$$A_{3}{}^{ijk,lmn} = \frac{1}{144}\sqrt{2}\,\varepsilon^{ijkpqr[lm}\,A_{2}{}^{n]}{}_{pqr}\,, \qquad (4.30)$$

and the new fermion variations are equal to

$$\begin{aligned} \delta_{g}\psi_{\mu}{}^{i} &= \sqrt{2}\,g\,A_{1}{}^{ij}\,\gamma_{\mu}\,\epsilon_{j}\,, \\ \delta_{g}\chi^{ijk} &= -2g\,A_{2l}{}^{ijk}\,\epsilon^{l}\,. \end{aligned} \qquad (4.31)$$

Here A_1 and A_2 are the components of the T-tensor defined in (3.28).

Furthermore we replace the abelian field strengths in the Lagrangian (4.26) by the field strengths $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, as described in subsection 2.3, and we include the topological and Chern-Simons-like terms (2.58). In the supersymmetry variations of the fermions we replace $F_{\mu\nu ij}$ accordingly by a tensor $\mathcal{H}_{\mu\nu ij}$ defined in analogy with (4.12),

$$\mathcal{H}^{+}_{\mu\nu\,ij} \equiv \mathcal{V}_{M\,ij}\,\mathcal{G}^{+\,M}_{\mu\nu}\,. \qquad (4.32)$$

Likewise we note three more relations,

$$\mathcal{V}_{M}{}^{ij}\,\mathcal{G}^{+\,M}_{\mu\nu} = -\frac{1}{2}\mathcal{O}^{+\,ij}_{\mu\nu}\,, \qquad (4.33)$$

$$\mathcal{G}^{+\,M}_{\mu\nu} = \mathrm{i}\Omega^{MN}\Big[\mathcal{V}_{N}{}^{ij}\mathcal{H}^{+}_{\mu\nu\,ij} + \frac{1}{2}\mathcal{V}_{N\,ij}\mathcal{O}^{+\,ij}_{\mu\nu}\Big]\,, \qquad (4.34)$$

$$\mathcal{H}^{+\,\Lambda}_{\mu\nu}\,\mathcal{O}^{+\mu\nu}_{\Lambda} = -\frac{1}{4}\mathcal{H}^{+}_{\mu\nu\,ij}\,\mathcal{O}^{+\mu\nu\,ij} + 2\,\mathcal{O}^{+}_{\mu\nu\Lambda}\,\mathcal{O}^{+\mu\nu}_{\Sigma}\,\mathcal{V}^{\Lambda}{}_{ij}\,\mathcal{V}^{\Sigma\,ij}\,, \qquad (4.35)$$

in direct analogy with (4.11), (4.13) and (4.16), respectively.

These above modifications generate a number of terms similar to (4.28) originating from the fermion variations proportional to $\mathcal{H}_{\mu\nu ij}$ in (4.29) and from the fermion variations (4.31) in the terms $\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{O}^{\mu\nu}{}_{\Lambda}$ in the original Lagrangian (4.26) (upon the replacement of the abelian field strengths by the $\mathcal{H}_{\mu\nu}{}^{\Lambda}$). Dropping terms of higher order in the fermions, these variations take the following form (here we also make use of (3.28)),

$$\delta\mathcal{L} = e\, g\, \mathcal{H}^{+}_{\mu\nu\, kl}\bigg[\frac{1}{3}T_i{}^{jkl}(\bar\epsilon^i\gamma^\rho\gamma^{\mu\nu}\psi_{\rho j} - \bar\epsilon_j\gamma^{\mu\nu}\gamma^\rho\psi_\rho{}^i) \\ + \frac{1}{72}\sqrt{2}\, T_{mnpq}{}^{kl}\, \varepsilon^{mnpqrstu}\, \bar\chi_{rst}\gamma^{\mu\nu}\epsilon_u\bigg] + \text{h.c.}\,. \tag{4.36}$$

Using the definition of the T-tensor (3.25) one can show that (4.36) and (4.28) combine to the expression,

$$\begin{aligned} \delta\mathcal{L} &= -e\, g\left[\mathcal{H}^{+}_{\mu\nu}{}^{M} - \mathcal{G}^{+}_{\mu\nu}{}^{M}\right] \qquad\qquad (4.37)\\ &\quad\times\left[\frac{1}{4}Q_{Mi}{}^{j}(\bar\epsilon^i\gamma^\rho\gamma^{\mu\nu}\psi_{\rho j} - \bar\epsilon_j\gamma^{\mu\nu}\gamma^\rho\psi_\rho{}^i) + \frac{1}{6}\sqrt{2}\,\mathcal{P}_M{}^{ijmn}\,\bar\epsilon_i\gamma_{\mu\nu}\chi_{jmn}\right] + \text{h.c.}\,, \end{aligned}$$

up to higher-order fermion terms. Here we made use of (4.34). For the electric components, where M is replaced by Λ, this term vanishes as one can read off from (2.53). The magnetic components can be cancelled by assigning a suitable supersymmetry variation to the tensor fields. Making use of (2.64) one can determine this variation directly,

$$\Theta^{\Lambda\alpha}\,\delta B_{\mu\nu\,\alpha} = \mathrm{i}\Big(\frac{2}{3}\sqrt{2}\,\mathcal{P}^{\Lambda}{}_{ijkl}\,\bar\epsilon^{[i}\,\gamma_{\mu\nu}\,\chi^{jkl]} + 4\,\mathcal{Q}^{\Lambda}{}_j{}^i\,\bar\epsilon_i\,\gamma_{[\mu}\,\psi_{\nu]}{}^j \;-\; \text{h.c.}\Big) \\ -2\,X^{\Lambda}{}_N{}^P\Omega_{PQ}\,A_{[\mu}{}^N\,\delta A_{\nu]}{}^Q\,. \tag{4.38}$$

At this point we have obtained a fairly complete version of all the supersymmetry transformations. In principle one can now continue and verify the cancellation of other variations of the Lagrangian. The pattern of cancellations is very similar to the pattern exhibited in [1]. In the following subsection we first summarize the full supersymmetry transformations and give the complete action. When comparing the results to those for the electric gaugings, the transformation rules for the magnetic gauge fields and the tensor fields do not enter. To verify the completeness of these transformation rules we will therefore verify the closure of the supersymmetry commutator for all the bosonic fields. This commutator will differ from (4.22), as there will be extra terms related to the gauge transformations and furthermore the shift transformation δ_{shift} is replaced by the tensor gauge transformations.

4.4 Gauged maximal supergravity

In this section we present the complete results for gauged supergravity. The supersymmetry transformation rules turn out to take the following form,

$$\begin{aligned} \delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i + \frac{1}{4}\sqrt{2}\,\hat{\mathcal{H}}^{-}_{\rho\sigma}{}^{ij}\,\gamma^{\rho\sigma}\gamma_\mu\epsilon_j + \frac{1}{4}\bar\chi^{ikl}\gamma^a\chi_{jkl}\,\gamma_a\gamma_\mu\epsilon^j \\ &\quad+\frac{1}{2}\sqrt{2}\,\bar\psi_{\mu k}\gamma^a\chi^{ijk}\,\gamma_a\epsilon_j - \frac{1}{576}\varepsilon^{ijklmnpq}\bar\chi_{klm}\gamma^{ab}\chi_{npq}\,\gamma_\mu\gamma_{ab}\epsilon_j \\ &\quad+\sqrt{2}\,g\,A_1{}^{ij}\,\gamma_\mu\,\epsilon_j\,, \end{aligned}$$

$$\begin{aligned}
\delta\chi^{ijk} &= -2\sqrt{2}\,\hat{\mathcal{P}}_\mu^{ijkl}\,\gamma^\mu\epsilon_l + \frac{3}{2}\,\hat{\mathcal{H}}^{-\,[ij}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{k]} - \frac{1}{24}\sqrt{2}\,\varepsilon^{ijklmnpq}\bar{\chi}_{lmn}\chi_{pqr}\,\epsilon^r \\
&\quad -2g\,A_{2l}{}^{ijk}\,\epsilon^l\;, \\
\delta e_\mu{}^a &= \bar{\epsilon}^i\gamma^a\psi_{\mu i} \;+\; \bar{\epsilon}_i\gamma^a\psi_\mu{}^i\;, \\
\delta\mathcal{V}_M{}^{ij} &= 2\sqrt{2}\,\mathcal{V}_{Mkl}\left(\bar{\epsilon}^{[i}\chi^{jkl]} + \frac{1}{24}\varepsilon^{ijklmnpq}\,\bar{\epsilon}_m\chi_{npq}\right), \\
\delta A_\mu{}^M &= -\mathrm{i}\,\Omega^{MN}\mathcal{V}_N{}^{ij}\left(\bar{\epsilon}^k\,\gamma_\mu\,\chi_{ijk} + 2\sqrt{2}\,\bar{\epsilon}_i\,\psi_{\mu j}\right) \;+\; \text{h.c.}\;, \\
\delta B_{\mu\nu\,\alpha} &= \frac{2}{3}\sqrt{2}\,t_{\alpha M}{}^P\Omega^{MQ}\left(\mathcal{V}_{P\,ij}\mathcal{V}_{Q\,kl}\,\bar{\epsilon}^{[i}\,\gamma_{\mu\nu}\,\chi^{jkl]} + 2\sqrt{2}\,\mathcal{V}_{P\,jk}\mathcal{V}_Q{}^{ik}\,\bar{\epsilon}_i\,\gamma_{[\mu}\,\psi_{\nu]}{}^j \;+\; \text{h.c.}\right) \\
&\quad - 2t_{\alpha\,M}{}^P\Omega_{PN}\,A_{[\mu}{}^M\,\delta A_{\nu]}{}^N\;. \qquad (4.39)
\end{aligned}$$

As was already noted before (see the text preceding (2.54)), we only need the variations $\Theta_M{}^\alpha\,\delta B_{\mu\nu\alpha}$, which can conveniently be written as,

$$\begin{aligned}
\Theta_M{}^\alpha\,\delta B_{\mu\nu\,\alpha} = \mathrm{i}\bigg(&\frac{2}{3}\sqrt{2}\,\mathcal{P}_{Mijkl}\,\bar{\epsilon}^{[i}\,\gamma_{\mu\nu}\,\chi^{jkl]} + 4\,\mathcal{Q}_{Mj}{}^i\,\bar{\epsilon}_i\,\gamma_{[\mu}\,\psi_{\nu]}{}^j \;-\; \text{h.c.}\bigg) \\
&- 2\,X_{MN}{}^P\Omega_{PQ}\,A_{[\mu}{}^N\,\delta A_{\nu]}{}^Q\;. \qquad (4.40)
\end{aligned}$$

The above variations were determined by the substitution of $\mathcal{H}^+_{\mu\nu ij}$ for $F^+_{\mu\nu ij}$ into (4.17) and by including the variations (4.31). For the tensor field $B_{\mu\nu\alpha}$ we based ourselves on (4.38).

At this point we return to the commutator of two supersymmetry transformations, which still takes the form (4.22), but now with the last 'shift' transformation on the magnetic gauge fields replaced by a full tensor gauge transformation,

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = \xi^\mu\hat{D}_\mu + \delta_{\rm L}(\epsilon^{ab}) + \delta_{\rm susy}(\epsilon_3) + \delta_{\rm SU(8)}(\Lambda^i{}_j) + \delta_{\rm gauge}(\Lambda^M) + \delta_{\rm tensor}(\Xi_{\mu\alpha})\,. \quad (4.41)$$

As before, the first term represents a covariantized general coordinate transformation, where one must now also include terms of order g induced by the gauging. The parameters ϵ_3 and Λ^M of the supersymmetry and gauge transformations appearing on the right-hand side, were already given in (4.24) and (4.25), respectively.

Because the magnetic vector and the tensor gauge fields are new as compared to previous treatments, we briefly consider the realization of (4.41) on the vector and tensor gauge fields. As a non-trivial consistency check on our reuslt, we include all higher-order fermion contributions in the supersymmetry commutator acting on the vector fields. For the tensor gauge field we include all bilinears in the fields χ^{ijk}. In this way we also determine the parameter of the tensor gauge transformation in (4.41). On the vector gauge fields we derive,

$$\begin{aligned}
[\delta(\epsilon_1),\delta(\epsilon_2)]A_\mu{}^M &= -\xi^\nu\left[\hat{\mathcal{G}}_{\mu\nu}{}^M - \frac{1}{2}\mathrm{i}\,\Omega^{MN}\mathcal{V}_N{}^{ij}\bar{\psi}_\mu{}^k\gamma_\nu\chi_{ijk} + \frac{1}{2}\mathrm{i}\,\Omega^{MN}\mathcal{V}_{Nij}\bar{\psi}_{\mu k}\gamma_\nu\chi^{ijk}\right] \\
&\quad + D_\mu\Lambda^M - gZ^{M,\alpha}\,\Xi_{\mu\alpha} + \delta(\epsilon_3)\,, \qquad (4.42)
\end{aligned}$$

where

$$\Theta_M{}^\alpha\,\Xi_{\mu\alpha} = -4\mathrm{i}\,\mathcal{Q}_{Mi}{}^j\,(\bar{\epsilon}_2{}^i\gamma_\mu\epsilon_{1j} + \bar{\epsilon}_{2j}\gamma_\mu\epsilon_1{}^i)\,, \qquad (4.43)$$

defines a contribution to the parameters of the tensor gauge transformations. These are not the only terms, as we will see by evaluating the first term on the right-hand side of (4.42). We remind the reader that we are only interested in the algebra acting on the fields $A_\mu{}^\Lambda$ and $\Theta^{\Lambda\alpha}A_{\mu\Lambda}$, as was explained at the end of subsection 2.3. This enables us to replace $\Theta^{\Lambda\alpha}\hat{\mathcal{G}}_{\mu\nu\Lambda}$ by $\Theta^{\Lambda\alpha}\hat{\mathcal{H}}_{\mu\nu\Lambda}$, by making use of the field equations (2.64) of the tensor field. This result applies also to the supercovariant extensions of the field strengths (this can be deduced from the observation that field equations transform into field equations under a symmetry of the action). Hence we must evaluate the expression,

$$
\begin{aligned}
-\,\xi^\nu\,\hat{\mathcal{H}}_{\mu\nu}{}^M &= \xi^\nu\partial_\nu A_\mu{}^M + \partial_\mu\xi^\nu A_\nu{}^M - D_\mu(\xi^\nu A_\nu{}^M) \\
&\quad - \mathrm{i}\xi^\nu\,\Omega^{MN}\Big[\mathcal{V}_N{}^{ij}[\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_{ijk} + \sqrt{2}\bar\psi_{\mu i}\psi_{\nu j}] - \text{h.c.}\Big] \\
&\quad - gZ^{M,\alpha}\,\xi^\nu\left[B_{\mu\nu\alpha} - t_{\alpha N}{}^Q\,\Omega_{PQ}\,A_\mu{}^N A_\nu{}^P\right]\,. \qquad (4.44)
\end{aligned}
$$

Combining this expression with the fermionic bilinears in (4.42) shows that the result decomposes into a space-time diffeomorphism with parameter ξ^μ, a nonabelian gauge transformation with parameter $-\xi^\mu A_\mu{}^M$, a supersymmetry transformation with parameter $-\frac{1}{2}\xi^\nu\psi_{\nu i}$, and a tensor gauge transformation with parameter $\xi^\nu(B_{\mu\nu\alpha} - t_{\alpha N}{}^Q\,\Omega_{PQ}\,A_\mu{}^N A_\nu{}^P)$.

Subsequently we evaluate the supersymmetry commutator on the tensor fields $\Theta_M{}^\alpha B_{\mu\nu\alpha}$. Including all terms quadratic in χ^{ijk}, we derive the following result,

$$
\begin{aligned}
[\delta(\epsilon_1),\delta(\epsilon_2)]\,\Theta_M{}^\alpha B_{\mu\nu\alpha} &= 2\,\Theta_M{}^\alpha\,D_{[\mu}\Xi_{\nu]\alpha} + 2\,X_{MN}{}^P\hat{\mathcal{G}}_{\mu\nu}{}^N\Omega_{PQ}\,\Lambda^Q \\
&\quad + \frac{2}{3}\mathrm{i}\sqrt{2}\left(\mathcal{P}_{Mijkl}\,\bar\epsilon_3^{[i}\,\gamma_{\mu\nu}\,\chi^{jkl]} - \text{h.c.}\right) \\
&\quad + \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\xi^\sigma\left[\frac{1}{3}\mathcal{P}_{M\,ijkl}\,\hat{\mathcal{P}}^{\rho\,ijkl} + \frac{1}{2}\mathcal{Q}_{Mi}{}^j\,\bar\chi^{ikl}\gamma^\rho\chi_{jkl}\right] \\
&\quad - 2\,X_{MN}{}^P\Omega_{PQ}\,A_{[\mu}{}^N\,[\delta(\epsilon_1),\delta(\epsilon_2)]A_{\nu]}{}^Q \;+\; \cdots\,, \qquad (4.45)
\end{aligned}
$$

where ξ^μ, Λ^M, ϵ_3 and $\Xi_{\mu\alpha}$ have already been given in (4.23), (4.25), (4.24) and (4.43), respectively, and the dots represent additional terms linear and quadratic in $\psi_\mu{}^i$. To derive this expression we used many of the results obtained previously. We draw attention to the fact that we also need the torsion constraint (4.21). Obviously the commutator closes with respect to these parameters in view of the fact that closure was already established on the gauge fields $A_\mu{}^M$. Note also the second term proportional to Λ^Q, which is implied by the last term shown in (2.54).

What remains is to investigate the closure relation for the terms proportional to the parameter ξ^μ of the general coordinate transformations. For these terms it is important to restrict ourselves to the commutator on $\Theta^{\Lambda\alpha}B_{\mu\nu\alpha}$, as these are the only components of the tensor field on which the supersymmetry algebra should be realized (we refer to the discussion at the end of subsection 2.3). We will first show that closure is indeed achieved provided the following equation holds,

$$
\frac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma\alpha}\,\Theta^{\Lambda\alpha} + eg\left(\frac{1}{3}\mathcal{P}^\Lambda{}_{ijkl}\,\hat{\mathcal{P}}^{\mu\,ijkl} + \frac{1}{2}\mathcal{Q}^\Lambda{}_i{}^j\,\bar\chi^{ikl}\gamma^\mu\chi_{jkl}\right) + \cdots \;=\; 0\,. \qquad (4.46)
$$

Here the unspecified terms are proportional to gravitino fields, which are suppressed throughout this calculation. This equation is simply the field equation associated with the magnetic vector fields (up to terms that vanish upon using the field equation for the tensor fields). The first term was aready evaluated in (2.66) and the second term originates from the minimal coulings which enter through $\mathcal{P}_M{}^{ijkl}$ and $\mathcal{Q}_{\mu i}{}^j$. It is perhaps unexpected that the supersymmetry algebra closes modulo a bosonic field equation that involves space-time derivatives, but one has to bear in mind that these particular field equations are only of first order in derivatives. Using the above equation we derive,

$$
\begin{aligned}
\mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\xi^\sigma\left[\frac{1}{3}\mathcal{P}^\Lambda{}_{ijkl}\,\hat{\mathcal{P}}^{\rho\,ijkl}+\frac{1}{2}\mathcal{Q}^\Lambda{}_i{}^j\,\bar\chi^{ikl}\gamma^\rho\chi_{jkl}\right] &= \Theta^{\Lambda\alpha}\left[\xi^\rho\partial_\rho B_{\mu\nu\alpha}-2\,\partial_{[\mu}\xi^\rho\,B_{\nu]\rho\alpha}\right]\\
&\quad +2\,\Theta^{\Lambda\alpha}D_{[\mu}\left[\xi^\rho(B_{\nu]\rho\alpha}-t_{\alpha N}{}^Q\,\Omega_{PQ}\,A_{\nu]}{}^N A_\rho{}^P)\right]-2\,X^\Lambda{}_N{}^P\,\mathcal{G}_{\mu\nu}{}^N\,\Omega_{PQ}\,\xi^\rho A_\rho{}^Q\\
&\quad +2\,X^\Lambda{}_N{}^P\Omega_{PQ}\,A_{[\mu}{}^N\left[\xi^\rho\partial_\rho A_{\nu]}{}^Q+\partial_{\nu]}\xi^\rho\,A_\rho{}^Q-2\,\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^Q\right]\,. \qquad (4.47)
\end{aligned}
$$

This establishes that full closure is indeed realized. The first line represents the required general coordinate transformation, the second and third term corresponds to the extra vector and tensor gauge transformations, respectively, with the same parameters as found in (4.44). Finally, the last term cancels against the similar terms generated on $A_\nu{}^Q$ by the commutator in the last term of (4.45). Here it is important to realize that this commutator does not fully close, in view of the fact that $A_\mu{}^Q$ includes all the magnetic gauge fields, as there is no contraction with $\Theta_Q{}^\alpha$. Nevertheless one is still left with a term proportional to $(\mathcal{G}-\mathcal{H})_{\nu\rho}{}^Q$, which can be absorbed into a transformation of type (2.67).

The full universal Lagrangian of maximal gauged supergravity in four space-time dimensions reads as follows,

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}e\,R-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\,(\bar\psi_\mu{}^i\gamma_\nu D_\rho\psi_{\sigma\,i}-\bar\psi_\mu{}^i\overleftarrow{D}_\rho\gamma_\nu\psi_{\sigma\,i})\\
&-\frac{1}{4}\,\mathrm{i}e\Big\{\mathcal{N}_{\Lambda\Sigma}\,\mathcal{H}^{+\,\Lambda}_{\mu\nu}\,\mathcal{H}^{+\mu\nu\Sigma}-\bar{\mathcal{N}}_{\Lambda\Sigma}\,\mathcal{H}^{-\,\Lambda}_{\mu\nu}\,\mathcal{H}^{-\mu\nu\Sigma}\Big\}\\
&-\frac{1}{12}e\,(\bar\chi^{ijk}\gamma^\mu D_\mu\chi_{ijk}-\bar\chi^{ijk}\overleftarrow{D}_\mu\gamma^\mu\chi_{ijk})-\frac{1}{12}e\,|\mathcal{P}^{ijkl}_\mu|^2\\
&-\frac{1}{12}\sqrt{2}\,e\left\{\bar\chi_{ijk}\gamma^\nu\gamma^\mu\psi_{\nu\,l}\,(\mathcal{P}^{ijkl}_\mu+\hat{\mathcal{P}}^{ijkl}_\mu)+\text{h.c.}\right\}\\
&+e\mathcal{H}^{+\Lambda}_{\mu\nu}\,\mathcal{O}^{+\mu\nu}_\Lambda+e\mathcal{H}^{-\Lambda}_{\mu\nu}\,\mathcal{O}^{-\mu\nu}_\Lambda-e\mathcal{V}^\Lambda{}_{ij}\,\mathcal{V}^{\Sigma ij}\left[\mathcal{O}^+_{\mu\nu\Lambda}\mathcal{O}^{+\mu\nu}_\Sigma+\mathcal{O}^-_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}_\Sigma\right]\\
&+\frac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{\Lambda\alpha}\,B_{\mu\nu\,\alpha}\left(2\partial_\rho A_{\sigma\,\Lambda}+gX_{MN\,\Lambda}\,A_\rho{}^M A_\sigma{}^N-\frac{1}{4}g\Theta_\Lambda{}^\beta B_{\rho\sigma\,\beta}\right)\\
&+\frac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN\,\Lambda}\,A_\mu{}^M A_\nu{}^N\left(\partial_\rho A_\sigma{}^\Lambda+\frac{1}{4}gX_{PQ}{}^\Lambda A_\rho{}^P A_\sigma{}^Q\right)\\
&+\frac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN}{}^\Lambda\,A_\mu{}^M A_\nu{}^N\left(\partial_\rho A_{\sigma\Lambda}+\frac{1}{4}gX_{PQ\Lambda}A_\rho{}^P A_\sigma{}^Q\right)\\
&+g\,e\Big\{\frac{1}{2}\sqrt{2}\,A_{1\,ij}\,\bar\psi^{\,i}_\mu\gamma^{\mu\nu}\psi^j_\nu+\frac{1}{6}A_{2i}{}^{jkl}\,\bar\psi^i_\mu\gamma^\mu\chi_{jkl}+A_3^{ijk,lmn}\,\bar\chi_{ijk}\,\chi_{lmn}+\text{h.c.}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- g^2 e \left\{ \frac{1}{24} A_{2i}{}^{jkl} A_2{}^{i}{}_{jkl} - \frac{3}{4} A_1^{ij} A_{1\,ij} \right\} \\
&+ \mathcal{L}_4 \,,
\end{aligned}
\tag{4.48}
$$

where $\mathcal{L}_4$ was given in (4.27). Here we included the scalar potential which appears at order g^2 and which takes the form already derived in [1]. We note that this potential can be written in various ways,

$$
\begin{aligned}
\mathcal{P}(\mathcal{V}) &= g^2 \left\{ \frac{1}{24} |A_{2i}{}^{jkl}|^2 - \frac{3}{4} |A_1^{ij}|^2 \right\} \\
&= \frac{1}{336}\, g^2\, \mathcal{M}^{MN} \left\{ 8\, \mathcal{P}_M{}^{ijkl} \mathcal{P}_{Nijkl} + 9\, \mathcal{Q}_{Mi}{}^{j}\, \mathcal{Q}_{Nj}{}^{i} \right\} \\
&= \frac{1}{672}\, g^2 \left\{ X_{MN}{}^{R} X_{PQ}{}^{S} \mathcal{M}^{MP} \mathcal{M}^{NQ} \mathcal{M}_{RS} + 7\, X_{MN}{}^{Q} X_{PQ}{}^{N} \mathcal{M}^{MP} \right\},
\end{aligned}
\tag{4.49}
$$

where we have used the real symmetric field-dependent 56×56 matrix $\mathcal{M}_{MN}$, defined by

$$
\mathcal{M}_{MN} \equiv \mathcal{V}_M{}^{ij}\, \mathcal{V}_{N\,ij} + \mathcal{V}_{M\,ij}\, \mathcal{V}_N{}^{ij} \,. \tag{4.50}
$$

Note that $\mathcal{M}$ is positive definite. Its inverse, $\mathcal{M}^{MN}$, can be written as

$$
\mathcal{M}^{MN} = \Omega^{MP} \Omega^{NQ} \mathcal{M}_{PQ} \,, \tag{4.51}
$$

by virtue of (3.4). This shows that $\det[\mathcal{M}] = 1$.

In the derivation of (4.49) we made use of the following equations,

$$
\begin{aligned}
X_{MN}{}^{R} X_{PQ}{}^{S} \mathcal{M}^{MP} \mathcal{M}^{NQ} \mathcal{M}_{RS} &= \mathcal{M}^{MN} \left(2\, \mathcal{P}_M{}^{ijkl} \mathcal{P}_{Nijkl} - 3\, \mathcal{Q}_{Mi}{}^{j}\, \mathcal{Q}_{Nj}{}^{i} \right) , \\
X_{MN}{}^{Q} X_{PQ}{}^{N} \mathcal{M}^{MP} &= \mathcal{M}^{MN} \left(2\, \mathcal{P}_M{}^{ijkl} \mathcal{P}_{Nijkl} + 3\, \mathcal{Q}_{Mi}{}^{j}\, \mathcal{Q}_{Nj}{}^{i} \right) , \\
\mathcal{M}^{MN} \mathcal{P}_M{}^{ijkl} \mathcal{P}_{Nijkl} &= 4\, |A_{2l}{}^{ijk}|^2 \,, \\
\mathcal{M}^{MN} \mathcal{Q}_{Mi}{}^{j}\, \mathcal{Q}_{Nj}{}^{i} &= -2\, |A_{2l}{}^{ijk}|^2 - 28\, |A_1{}^{ij}|^2 \,,
\end{aligned}
\tag{4.52}
$$

which can be derived using various results and definitions presented in section 3.

5. Discussion and applications

In this paper we have presented the complete construction of all gaugings of four-dimensional maximal supergravity. We have shown that gaugings can be completely characterized in terms of an embedding tensor, subject to a linear and a quadratic constraint, (2.12) and (2.9), respectively. A generic gauging can involve both electric and magnetic charges, together with two-form tensor fields transforming in the **133** representation of $E_{7(7)}$. The addition of magnetic vector fields and the two-rank tensor fields does not lead to additional degrees of freedom owing to the presence of extra gauge invariances associated with these fields. We have presented the full Lagrangian of the theory in (4.48) and the supersymmetry transformations in (4.39).

In this last section we briefly demonstrate the group-theoretical approach of this paper to construct actual gaugings of maximal supergravity in four dimensions. The starting point is the construction of a solution to the constraints (2.12) and (2.9) on the embedding tensor $\Theta_M{}^\alpha$. The former one is a linear constraint whose general solution is explicitly known as the 912-dimensional image of a projector. The most straightforward strategy will thus be to start from a particular solution to this constraint and impose on it the quadratic constraint.

Of course, when one wants to see if a specific subgroup of $\mathrm{E}_{7(7)}$ can be gauged, it suffices to simply verify whether the constraints are satisfied on the corresponding embedding tensor. In other cases, when one wants to explore a variety of gaugings, it is often useful to first select a subgroup $\mathrm{G}_0 \subset \mathrm{E}_{7(7)}$ in which the gauge group will be embedded eventually. This group may be a manifest invariance of the ungauged Lagrangian in a suitable electric/magnetic duality frame. When this is the case, the gauging will only involve electric gauge fields and there is no need for introducing dual vector and tensor fields. Branching the **912** of $\mathrm{E}_{7(7)}$ under G_0 and scanning through the different irreducible components allows a systematic study of the quadratic constraint (2.9) and thereby a full determination of the corresponding admissible gaugings. In that case the closure of the gauge group is already guaranteed, owing to the equivalent formulation (2.16) of the quadratic constraint, so that every solution to the linear constraint (2.12) will define a viable gauging.

A central result of this paper is that it is not necessary to restrict G_0 to a group that can be realized as an invariance of the ungauged Lagrangian that serves as a starting point for the gauging. In that case, one must simply analyze both constraints and the gauging may eventualy comprise both electric and magnetic charges. It is important to realize that the scalar potential is insensitive to the issue of electric/magnetic frames, so that its stationary point can be directly studied. Scanning through the different choices of G_0, it is straightforward to construct the various corresponding gaugings of the four-dimensional theory. In the following we will illustrate the strategy by first reproducing the known gaugings, subsequently sketching the construction of gaugings related to flux compactifications of IIA and IIB supergravity and finally giving some other examples, including Scherk-Schwarz reductions from higher dimensions.

5.1 Known gaugings

As first examples, let us briefly review the known gaugings embedded in the groups $\mathrm{G}_0 = \mathrm{SL}(8,\mathbb{R})$ and $\mathrm{G}_0 = \mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$, respectively. It is known that there are corresponding ungauged Lagrangians which have these groups as an invariance group. Hence we can restrict ourselves to analyzing the linear constriant. With the group $\mathrm{G}_0 = \mathrm{SL}(8,\mathbb{R})$, the branching of the $\mathrm{E}_{7(7)}$ representations associated with the vector fields, the adjoint representation and the embedding tensor, is as follows,

$$\begin{aligned} \mathbf{56} &\rightarrow \mathbf{28} + \mathbf{28}' \,, \\ \mathbf{133} &\rightarrow \mathbf{63} + \mathbf{70} \,, \\ \mathbf{912} &\rightarrow \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}' \,, \end{aligned} \tag{5.1}$$

where the **28** representation in the first decomposition denotes the electric gauge fields. The possible couplings between vector fields and $\mathrm{E}_{7(7)}$ symmetry generators induced by the various Θ components according to (2.1) can be summarized in the table

$$\begin{array}{c|cc} & \mathbf{28} & \mathbf{28}' \\ \hline \mathbf{63} & \mathbf{36}+\mathbf{420} & \mathbf{36}'+\mathbf{420}' \\ \mathbf{70} & \mathbf{420}' & \mathbf{420} \end{array} \tag{5.2}$$

where the left column represents the $\mathrm{E}_{7(7)}$ generators, and the top row represents the vector fields. The entries correspond to the *conjugate* representations of the respective components of the embedding tensor belonging to the **912** representation. Restricting to gaugings embedded into $\mathrm{G}_0 = \mathrm{SL}(8,\mathbb{R})$, the upper left entry is relevant. However, the **420** would alsocouple to the magnetic gauge fields and the remaining generators of $\mathrm{E}_{7(7)}$ so that the embedding tensor is restricted to live in the $\mathbf{36}'$ (i.e. the *conjugate* of the **36** indicated in the table). Every element in the $\mathbf{36}'$ defines a viable gauging. A closer analysis shows [7] that modulo $\mathrm{SL}(8,\mathbb{R})$ conjugation the general form of $\Theta \in \mathbf{36}'$ is given by

$$\Theta_M{}^\alpha = \Theta_{[AB]}{}^C{}_D = \delta^C_{[A}\,\theta_{B]D}\;, \qquad \theta_{AB} = \mathrm{diag}\{\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q},\underbrace{0,\dots,0}_{r}\}\;, \tag{5.3}$$

with $A, B = 1,\dots,8$, and reproduces the $\mathrm{CSO}(p,q,r)$ gaugings [2, 27], where $p+q+r = 8$. There are 24 inequivalent gaugings of this type.

Choosing the group $\mathrm{G}_0 = \mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$, which is group that can be used to identify gaugings that are related to compactifications from five dimensions, the branchings of the three relevant representations are,

$$\begin{aligned} \mathbf{56} &\to \mathbf{1}_{-3} + \overline{\mathbf{27}}_{-1} + \mathbf{27}_{+1} + \mathbf{1}_{+3}\,, \\ \mathbf{133} &\to \mathbf{27}_{-2} + \mathbf{1}_0 + \mathbf{78}_0 + \overline{\mathbf{27}}_{+2}\,, \\ \mathbf{912} &\to \mathbf{78}_{-3} + \overline{\mathbf{27}}_{-1} + \overline{\mathbf{351}}_{-1} + \mathbf{351}_{+1} + \mathbf{27}_{+1} + \mathbf{78}_{+3}\,, \end{aligned} \tag{5.4}$$

The first decomposition again captures the split into electric and magnetic vector fields with the graviphoton transforming in the $\mathbf{1}_{-3}$ and the 27 gauge fields from the five-dimensional theory in the $\overline{\mathbf{27}}_{-1}$ representation. The couplings between vector fields and $\mathrm{E}_{7(7)}$ symmetry generators induced by the various Θ components can be summarized in a table analogous to (5.2),

$$\begin{array}{c|cccc} & \mathbf{1}_{-3} & \overline{\mathbf{27}}_{-1} & \mathbf{27}_{+1} & \mathbf{1}_{+3} \\ \hline \mathbf{27}_{-2} & & \mathbf{78}_{-3} & \overline{\mathbf{351}}_{-1}+\overline{\mathbf{27}}_{-1} & \mathbf{27}_{+1} \\ \mathbf{78}_0 & \mathbf{78}_{-3} & \overline{\mathbf{351}}_{-1}+\overline{\mathbf{27}}_{-1} & \mathbf{351}_{+1}+\mathbf{27}_{+1} & \mathbf{78}_{+3} \\ \mathbf{1}_0 & & \overline{\mathbf{27}}_{-1} & \mathbf{27}_{+1} & \\ \overline{\mathbf{27}}_{+2} & \overline{\mathbf{27}}_{-1} & \mathbf{351}_{+1}+\mathbf{27}_{+1} & \mathbf{78}_{+3} & \end{array} \tag{5.5}$$

The table shows that a gauging involving only electric vector fields can only live in the $\mathbf{78}_{+3}$ representation. Vice versa, every such embedding tensor automatically satisfies the quadratic constraint (2.16) and thus defines a viable gauging. These are the theories descending from five dimensions by Scherk-Schwarz reduction [28, 3, 7].

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5.2 Flux gaugings

Here we consider gaugings of $N = 8$ supergravity that can in principle be generated by (generalized) toroidal flux compactifications of type-IIB and M-theory. The proper setting to discuss these theories is a decomposition $\mathrm{E}_{7(7)}$ group according to $\mathrm{SL}(2) \times \mathrm{SL}(6)$ and $\mathrm{SL}(7)$, respectively. For the type-IIB theory this embedding is realized as

$$\mathrm{E}_{7(7)} \longrightarrow \mathrm{SL}(6) \times \mathrm{SL}(3) \longrightarrow \mathrm{SL}(6) \times \mathrm{SL}(2) \times \mathrm{SO}(1,1) \,. \tag{5.6}$$

The S-duality group coincides with the SL(2) factor. Electric and magnetic charges transform according to the **56** representation which branches as

$$\mathbf{56} \to (\mathbf{6}',\mathbf{1})_{-2} + (\mathbf{6},\mathbf{2})_{-1} + (\mathbf{20},\mathbf{1})_0 + (\mathbf{6}',\mathbf{2})_{+1} + (\mathbf{6},\mathbf{1})_{+2} \,.$$

Here, the $(\mathbf{6}',\mathbf{1})_{-2}$ and $(\mathbf{6},\mathbf{2})_{-1}$ representations descend from graviphotons and two-forms, respectively, while the four-form generates gauge fields which, together with their magnetic duals, constitute the $(\mathbf{20},\mathbf{1})_0$. The couplings between vector fields and $\mathrm{E}_{7(7)}$ symmetry generators is summarized in the following table [9],

	$(\mathbf{6}',\mathbf{1})_{-2}$	$(\mathbf{6},\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{1})_0$	$(\mathbf{6}',\mathbf{2})_{+1}$	$(\mathbf{6},\mathbf{1})_{+2}$
$(\mathbf{1},\mathbf{2})_{-3}$		$(\mathbf{6},\mathbf{1})_{-4}$	$(\mathbf{20},\mathbf{2})_{-3}$	$(\mathbf{6}',\mathbf{3}+\mathbf{1})_{-2}$	$(\mathbf{6},\mathbf{2})_{-1}$
$(\mathbf{15},\mathbf{1})_{-2}$	$(\mathbf{6},\mathbf{1})_{-4}$	$(\mathbf{20},\mathbf{2})_{-3}$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{1})_{-2}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{2})_{-1}$	$(\mathbf{70}{+}\mathbf{20},\mathbf{1})_0$
$(\mathbf{15}',\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{2})_{-3}$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{1})_{-2} + (\mathbf{6}',\mathbf{3})_{-2}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{3}{+}\mathbf{1})_0 + (\mathbf{70}',\mathbf{1})_0$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{2})_{+1}$
$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{6}',\mathbf{1})_{-2}$	$(\mathbf{6},\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{1})_0$	$(\mathbf{6}',\mathbf{2})_{+1}$	$(\mathbf{6},\mathbf{1})_{+2}$
$(\mathbf{35},\mathbf{1})_0$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{1})_{-2}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{2})_{-1}$	$(\mathbf{70}{+}\mathbf{70}'{+}\mathbf{20},\mathbf{1})_0$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{2})_{+1}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{1})_{+2}$
$(\mathbf{1},\mathbf{3})_0$	$(\mathbf{6}',\mathbf{3})_{-2}$	$(\mathbf{6},\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{3})_0$	$(\mathbf{6}',\mathbf{2})_{+1}$	$(\mathbf{6},\mathbf{3})_{+2}$
$(\mathbf{15},\mathbf{2})_{+1}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{2})_{-1}$	$(\mathbf{20},\mathbf{3}{+}\mathbf{1})_0 + (\mathbf{70},\mathbf{1})_0$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{2})_{+1}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{1})_{+2} + (\mathbf{6},\mathbf{3})_{+2}$	$(\mathbf{20},\mathbf{2})_{+3}$
$(\mathbf{15}',\mathbf{1})_{+2}$	$(\mathbf{70}'{+}\mathbf{20},\mathbf{1})_0$	$(\mathbf{6}'{+}\mathbf{84}',\mathbf{2})_{+1}$	$(\mathbf{6}{+}\mathbf{84},\mathbf{1})_{+2}$	$(\mathbf{20},\mathbf{2})_{+3}$	$(\mathbf{6}',\mathbf{1})_{+4}$
$(\mathbf{1},\mathbf{2})_{+3}$	$(\mathbf{6}',\mathbf{2})_{+1}$	$(\mathbf{6},\mathbf{3}{+}\mathbf{1})_{+2}$	$(\mathbf{20},\mathbf{2})_{+3}$	$(\mathbf{6}',\mathbf{1})_{+4}$	

The entries of the table correspond to the various conjugate representations of the respective components of the embedding tensor. Within the **912** all these components appear with multiplicity 1 apart from the $(\mathbf{6},\mathbf{2})_{-1}$ and $(\mathbf{6}',\mathbf{2})_{+1}$ which appear with multiplicity 2. It follows from the table that an embedding tensor in the $(\mathbf{6}',\mathbf{1})_{+4}$ defines a purely electric gauging which thus automatically satisfies the quadratic constraint. This corresponds to the theory induced by a five-form flux. A three-form flux on the other hand induces a component of the embedding tensor in the $(\mathbf{20},\mathbf{2})_{+3}$ represention, which involves electric and magnetic gauge fields in the $(\mathbf{20},\mathbf{1})_0$. Consistency thus requires to further impose the quadratic constraint (2.16) onto Θ, leading to [9]

$$\varepsilon^{\Lambda\Sigma\Gamma\Omega\Pi\Delta}\, \theta_{\Lambda\Sigma\Gamma}{}^{\sigma}\, \theta_{\Omega\Pi\Delta}{}^{\tau} = 0 \,, \tag{5.7}$$

with $\sigma,\tau = 1,2$, $\Lambda,\Sigma = 1,\ldots,6$. Here $\theta_{\Lambda\Sigma\Gamma}{}^{\tau}$ denotes the components of the embedding tensor corresponding to the $(\mathbf{20},\mathbf{2})_{+3}$ representation. The above constraint is precisely the tadpole cancellation condition on the NS-NS and R-R 3-form fluxes.

Gaugings defined by Θ-components with lower $\mathrm{SO}(1,1)$ grading will correspond to the theories induced by geometric fluxes (twists), non-geometric compactifications, etc. It follows from the table that the quadratic constraint (2.16) leads to more and more consistency conditions among these lower Θ-components as they tend to stronger mix electric and magnetic vector fields. It is, however, straightforward to work out these constraints by branching (2.16) accordingly (recall also that the total representation content of this constraint is given by the $\mathbf{133}+\mathbf{8645}$) of $\mathrm{E}_{7(7)}$. Another representation in the above table which is relevant to string compactifications is the $(\mathbf{84},\mathbf{1})_{+2}$. It corresponds to the geometric flux $\tau_{\Lambda\Sigma}{}^{\Gamma}$ which describes a "twisted" six-torus. The quadratic constraint implies the following condition,

$$\tau_{[\Lambda\Sigma}{}^{\Gamma}\,\tau_{\Pi]\Gamma}{}^{\Delta}=0\,. \tag{5.8}$$

One may wonder which components of the embedding tensor describe the non-geometric-fluxes $Q_{\Lambda}{}^{\Sigma\Gamma}$ and $R^{\Lambda\Sigma\Gamma}$ obtained from $\tau_{\Lambda\Sigma}{}^{\Gamma}$ by applying two subsequent T-dualities along the directions Σ and Λ, respectively [29, 30]. Using the flux/weight correspondence defined in [31] we can identify these non-geometric fluxes with the following representations:

$$Q_{\Lambda}{}^{\Sigma\Gamma}\in(\mathbf{84}',\mathbf{2})_{+1}\quad;\qquad R^{\Lambda\Sigma\Gamma}\in(\mathbf{20},\mathbf{3})_{0}\,. \tag{5.9}$$

We notice that T-duality changes the $\mathrm{SL}(2,\mathbb{R})$ representation of the flux on which it acts. We leave a detailed analysis for future work.

A similar analysis of M-theory fluxes has been performed in [32], see also [33, 34]. In this case the relevant embedding of the torus GL(7) is according to

$$\mathrm{E}_{7(7)}\;\longrightarrow\;\mathrm{SL}(8)\;\longrightarrow\;\mathrm{SL}(7)\times\mathrm{SO}(1,1)\,. \tag{5.10}$$

Electric and magnetic charges transform according to the branching

$$\mathbf{56}\to\mathbf{7}'_{-3}+\mathbf{21}_{-1}+\mathbf{21}'_{+1}+\mathbf{7}_{+3}\,, \tag{5.11}$$

where the $\mathbf{7}'_{-3}$ and the $\mathbf{21}_{-1}$ descend from graviphotons and antisymmetric tensors, respectively. The couplings between vector fields and $\mathrm{E}_{7(7)}$ symmetry generators are given as [32],

	$\mathbf{7}'_{-3}$	$\mathbf{21}_{-1}$	$\mathbf{21}'_{+1}$	$\mathbf{7}_{+3}$
$\mathbf{7}_{-4}$	$\mathbf{1}_{-7}$	$\mathbf{35}_{-5}$	$(\mathbf{140}'+\mathbf{7}')_{-3}$	$(\mathbf{28}+\mathbf{21})_{-1}$
$\mathbf{35}'_{-2}$	$\mathbf{35}_{-5}$	$\mathbf{140}'_{-3}$	$(\mathbf{21}+\mathbf{224})_{-1}$	$(\mathbf{21}'+\mathbf{224}')_{+1}$
$\mathbf{48}_{0}$	$(\mathbf{140}'+\mathbf{7}')_{-3}$	$(\mathbf{21}+\mathbf{28}+\mathbf{224})_{-1}$	$(\mathbf{21}'+\mathbf{28}'+\mathbf{224}')_{+1}$	$(\mathbf{140}+\mathbf{7})_{+3}$
$\mathbf{1}_{0}$	$\mathbf{7}'_{-3}$	$\mathbf{21}_{-1}$	$\mathbf{21}'_{+1}$	$\mathbf{7}_{+3}$
$\mathbf{35}_{+2}$	$(\mathbf{21}+\mathbf{224})_{-1}$	$(\mathbf{21}'+\mathbf{224}')_{+1}$	$\mathbf{140}_{+3}$	$\mathbf{35}'_{+5}$
$\mathbf{7}'_{+4}$	$(\mathbf{28}'+\mathbf{21}')_{+1}$	$(\mathbf{140}+\mathbf{7})_{+3}$	$\mathbf{35}'_{+5}$	$\mathbf{1}_{+7}$

The table shows that an embedding tensor in the $\mathbf{1}_{+7}$ and in the $\mathbf{35}'_{+5}$ representation define electric gaugings that automatically satisfy the quadratic constraint. They describe the theories obtained by switching on in eleven dimensions a seven-form g_7 and a four-form flux $g_{\Lambda\Sigma\Gamma\Delta}$, respectively. The former theory has in fact already been considered in [35]. An embedding tensor in the $\mathbf{140}_{+3}$ corresponds to the parameters $\tau_{\Lambda\Sigma}{}^{\Xi}$ of a geometric flux and is subject to the quadratic constraint (2.16),

$$\tau_{[\Lambda\Sigma}{}^{\Omega}\tau_{\Gamma]\Omega}{}^{\Xi}=0\,, \tag{5.12}$$

with $\Lambda, \Sigma = 1, \ldots, 7$, corresponding to the Jacobi identity of the associated gauge algebra. If g_7, $g_{\Lambda\Sigma\Gamma\Delta}$ and the geometric flux $\tau_{\Lambda\Sigma}{}^{\Gamma}$ are switched on together, the second order constraint on the embedding tensor, as was shown in [32], yields the additional condition

$$\tau_{[\Lambda\Sigma}{}^{\Delta}\, g_{\Gamma\Pi\Omega]\Delta} = 0\,, \tag{5.13}$$

originally found in [33].

5.3 Gaugings of six-dimensional origin

In this subsection we demonstrate our method for gaugings that arise, in particular, from a two-fold Scherk-Schwarz reduction from six space-time dimensions [4, 5]. The Scherk-Schwarz reduction of maximal supergravity from $D = 5$ to $D = 4$ spacetime dimensions was first constructed in [28] and recently this theory was obtained more directly in four spacetime dimensions by gauging [3]. For a general treatment of Scherk-Schwarz reductions in relation to gauged maximal supergravities, see [7], where the Scherk-Schwarz reduced maximal supergravity from $D = 6$ to $D = 5$ was constructed as a gauging of five-dimensional supergravity.

The proper choice for G_0 is the maximal subgroup

$$\mathrm{E}_{7(7)} \;\longrightarrow\; \mathrm{SO}(5,5) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(1,1)\;, \tag{5.14}$$

where $\mathrm{SO}(5,5)$ represents the non-linear symmetry group of maximal supergravity in six dimensions and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(1,1)$ is the group corresponding to the reduction on a two-torus. Electric and magnetic charges branch as

$$\mathbf{56} \rightarrow (\mathbf{1},\mathbf{2})_{-2} + (\overline{\mathbf{16}},\mathbf{1})_{-1} + (\mathbf{10},\mathbf{2})_{0} + (\mathbf{16},\mathbf{1})_{+1} + (\mathbf{1},\mathbf{2})_{+2}\;, \tag{5.15}$$

where the $(\mathbf{1},\mathbf{2})_{-2}$ and $(\overline{\mathbf{16}},\mathbf{1})_{-1}$ correspond to graviphotons and six-dimensional vectors, respectively, while the $(\mathbf{10},\mathbf{2})_0$ combines the electric vectors and their magnetic duals descending from the self-dual two-forms in six-dimensions. Their couplings to the $\mathrm{E}_{7(7)}$ symmetry generators are summarized in the table below,

	$(\mathbf{1},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}},\mathbf{1})_{-1}$	$(\mathbf{10},\mathbf{2})_{0}$	$(\mathbf{16},\mathbf{1})_{+1}$	$(\mathbf{1},\mathbf{2})_{+2}$
$(\mathbf{10},\mathbf{1})_{-2}$		$(\mathbf{16},\mathbf{1})_{-3}$	$(\mathbf{1}+\mathbf{45},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}}+\mathbf{144},\mathbf{1})_{-1}$	$(\mathbf{10},\mathbf{2})_{0}$
$(\mathbf{16},\mathbf{2})_{-1}$	$(\mathbf{16},\mathbf{1})_{-3}$	$(\mathbf{1}+\mathbf{45},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}},\mathbf{3}+\mathbf{1})_{-1}+(\mathbf{144},\mathbf{1})_{-1}$	$(\mathbf{10}+\mathbf{120},\mathbf{2})_{0}$	$(\mathbf{16},\mathbf{3}+\mathbf{1})_{+1}$
$(\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}},\mathbf{3})_{-1}$	$(\mathbf{10},\mathbf{2})_{0}$	$(\mathbf{16},\mathbf{3})_{+1}$	$(\mathbf{1},\mathbf{2})_{+2}$
$(\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}},\mathbf{1})_{-1}$	$(\mathbf{10},\mathbf{2})_{0}$	$(\mathbf{16},\mathbf{1})_{+1}$	$(\mathbf{1},\mathbf{2})_{+2}$
$(\mathbf{45},\mathbf{1})_{0}$	$(\mathbf{45},\mathbf{2})_{-2}$	$(\overline{\mathbf{16}}+\mathbf{144},\mathbf{1})_{-1}$	$(\mathbf{10}+\mathbf{120},\mathbf{2})_{0}$	$(\mathbf{16}+\overline{\mathbf{144}},\mathbf{1})_{+1}$	$(\mathbf{45},\mathbf{2})_{+2}$
$(\overline{\mathbf{16}},\mathbf{2})_{+1}$	$(\overline{\mathbf{16}},\mathbf{3}+\mathbf{1})_{-1}$	$(\mathbf{10}+\mathbf{120},\mathbf{2})_{0}$	$(\mathbf{16},\mathbf{3}+\mathbf{1})_{+1}+(\overline{\mathbf{144}},\mathbf{1})_{+1}$	$(\mathbf{1}+\mathbf{45},\mathbf{2})_{+2}$	$(\overline{\mathbf{16}},\mathbf{1})_{+3}$
$(\mathbf{10},\mathbf{1})_{+2}$	$(\mathbf{10},\mathbf{2})_{0}$	$(\mathbf{16}+\overline{\mathbf{144}},\mathbf{1})_{+1}$	$(\mathbf{1}+\mathbf{45},\mathbf{2})_{+2}$	$(\overline{\mathbf{16}},\mathbf{1})_{+3}$	

This shows that an embedding tensor in the $(\overline{\mathbf{16}},\mathbf{1})_{+3}$ defines a consistent electric gauging corresponding to the theory obtained by giving a T^2 flux to the six-dimensional vector field strength. A Scherk-Schwarz gauging is defined by an embedding tensor in the $(\mathbf{45},\mathbf{2})_{+2}$, i.e. a tensor of type $\theta_{u,mn}$ with $m, n = 1, \ldots, 10$, $u = 1, 2$. This corresponds to identifying the two gauge group generators $X_u = \theta_{u,mn}\, t^{mn}$ generating a subgroup of $\mathrm{SO}(5,5)$ associated with the dependence of the fields on the internal T^2 according to the Scherk-Schwarz ansatz, which couple to the two graviphotons. As this gauging a priori involves electric and magnetic vector fields, the quadratic constraint (2.16) poses a nontrivial restriction,

$$\epsilon^{uw}\,\eta^{mn}\,\theta_{u,mp}\,\theta_{w,nq} = 0\,, \tag{5.16}$$

which implies $[X_u, X_v] = 0$. This is consistent as these generators must commute in the multiple Scherk-Schwarz reduction. The complete gauge algebra in four dimensions takes the form

$$\begin{aligned} [X_u,\, X_v] = 0\,, \qquad & [X_u,\, X_\sigma] \;\propto\; \theta_{u,mn}(\Gamma^{mn})_\sigma{}^\tau\, X_\tau\,, \\ [X_u,\, X_{mw}] \propto \theta_{u,mn}\,\eta^{np}\, X_{pw}\,, \qquad & [X_\sigma,\, X_\tau] \;\propto\; \epsilon^{uv}\theta_{u,mn}(\Gamma^{mnp})_{\sigma\tau}\, X_{pv}\,, \end{aligned} \tag{5.17}$$

with SO(5,5) Γ-matrices $\Gamma^m_{\sigma\tau}$. We intend to give a detailed analysis of this theory elsewhere.

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